(10) Combined Mathematics

Structure of the Question Paper

Paper I - **Time : 03 hours** (In addition, 10 minutes for reading.)

This paper consists of **two** parts.

Part A : Ten questions. **All** questions should be answered. 25 marks for each question - altogether 250 marks.

Part B : Seven questions. **Five** questions should be answered. Each question carries 150 marks - altogether 750 marks.

Total marks for paper I $1000 \div 10 = 100$

Paper II - **Time : 03 hours** (In addition, 10 minutes for reading.)

This paper consists of **two** parts.

Part A : Ten questions. **All** questions should be answered. 25 marks for each question - altogether 250 marks.

Part B : Seven questions. **Five** questions should be answered. Each question carries 150 marks - altogether 750 marks.

Total marks for paper II $1000 \div 10 = 100$

Calculation of the final mark :

Paper I	=	100
Paper II	=	100
Final mark	=	$200 \div 2 = \underline{\underline{100}}$

(10) Combined Mathematics

Paper I

Part A

1. Using the **Principle of Mathematical Induction**, prove that $6^n - 1$ is divisible by 5 for all $n \in \mathbb{Z}^+$.

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2. Find the set of all real values of x satisfying the inequality $2|x - 3| \leq 2 + x$.

Hence, solve $2|x + 3| \leq 2 - x$.

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3. Shade the region R that represents the complex numbers z satisfying the conditions $|z - i| \leq 1$ and $\frac{\pi}{4} \leq \operatorname{Arg}(z - i) \leq \frac{3\pi}{4}$ in an Argand diagram.

Write down the maximum value of $\operatorname{Re} z + \operatorname{Im} z$ for z in the region R.

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4. Show that $\lim\limits_{x \to 0} \dfrac{\left((8 + x)^{\frac{1}{3}} - 2\right) \sin 2x}{x^2} = \dfrac{1}{6}$.

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5. Show that the equation of the tangent to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ at the point $P \equiv (4\cos\theta, 3\sin\theta)$ is $\frac{x}{4}\cos\theta + \frac{y}{3}\sin\theta = 1$.

Find the value θ $(0 < \theta < \frac{\pi}{2})$ such that the normal to the above ellipse at P passes through the point $\left(0, -\frac{7}{6}\right)$.

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6. Differentiate $\tan^{-1}\left[\frac{5}{3}\tan\left(\frac{x}{2}\right) + \frac{4}{3}\right]$ with respect to x. **Hence,** find $\int \frac{dx}{5 + 4\sin x}$.

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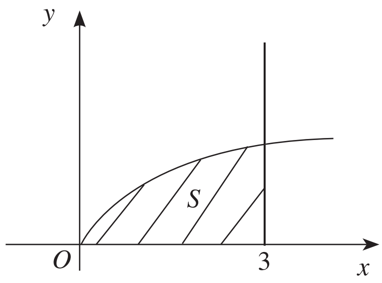
7. Let S be the region bounded by the curve $y = \frac{x}{\sqrt{x^2+9}}$, the straight line $x = 3$ and the x–axis (see the figure). Show that the volume of the solid generated by rotating S about the x–axis through 2π radians is $3\pi\left(1 - \frac{\pi}{4}\right)$.



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8. A variable straight line through the point (2, 1) meets the x–axis and the y–axis at the points P and Q respectively. The point R is the mid-point of PQ. Show that the point R lies on the curve $x + 2y = 2xy$.

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9. Find the equation of the circle through the points $(0, 0)$ and $(0, 2)$ which bisects the circumference of the circle $x^2 + y^2 - 2x + 4y - 6 = 0$.

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10. Express $\sqrt{3}\ \cos x - \sin x$ in the form $R\cos(x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

Hence, solve the equation $\sqrt{3}\ \cos 2x - \sin 2x + 1 = 0$.

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Part B

11. (*a*) Let a and b be two distinct real numbers. Show that the roots of the equation $x^2 + 2bx + 2ab = a^2$ are real and distinct.

Show that the roots of the above equation α and β are both non-zero if and only if $a \neq 2b$ and $a \neq 0$.

Now suppose that $a \neq 2b$ and $a \neq 0$. Find the quadratic equation with $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots.

(*b*) Let $f(x)$ be a polynomial of degree greater than 2, and let p and q be distinct real numbers. Applying the Remainder Theorem twice, show that the remainder when $f(x)$ is divided by $(x-p)(x-q)$ is $\frac{f(q)-f(p)}{q-p}(x-p)+f(p)$.

Let $g(x) = x^3 + ax^2 + bx + 1$, where $a, b \in \mathbb{R}$. It is given that the remainder when $g(x)$ is divided by $(x-2)$ is thrice the remainder when it is divided by $(x-1)$, and that the remainder when $g(x)$ is divided by $(x-1)(x-2)$ is $kx+5$, where $k \in \mathbb{R}$. Find the values of a, b and k.

12. (*a*) Show that the term independent of x in the expansion of $(1+x)^2\left(2x^2 - \frac{1}{2x}\right)^{10}$ is -15.

(*b*) A relay team consisting of 4 sprinters is to be selected from among 8 sprinters of different performance records. If the least performer among them is selected, then the best performer is also selected; but the best performer can be selected without the least performer being selected. Find the number of different relay teams that can be formed.

(*c*) Let $u_r = \frac{2r^2-5}{(r+1)^2(r+2)^2}$ and $f(r) = \frac{\lambda r + \mu}{(r+1)^2}$ for $r \in \mathbb{Z}^+$, where λ and μ are real constants.

Find the values of λ and μ such that $u_r = f(r) - f(r+1)$ for $r \in \mathbb{Z}^+$.

Let $S_n = \sum_{r=1}^{n} u_r$ for $n \in \mathbb{Z}^+$. Show that $S_n = \frac{1}{4} - \frac{(2n+1)}{(n+2)^2}$ for $n \in \mathbb{Z}^+$.

Deduce that the infinite series $\sum_{r=1}^{\infty} u_r$ is convergent and find its sum.

13. (*a*) Let $a, b, c \in \mathbb{R}$. Also, let $A = \begin{pmatrix} 1 & 2 & 1 \\ a & 3 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & b & 1 \\ b & 1 & c \end{pmatrix}$ and $C = \begin{pmatrix} c & 2a+c \\ 1 & b \end{pmatrix}$.

Find the values of a, b and c such that $AB^T = C$.

For these values of a, b and c, find $(C^T)^{-1}$ and **hence**, find the matrix P such that $C^{-1} P C^T = 5C$.

(*b*) Using de Moivre's Theorem for a positive intergral index, show that if $z = \cos\theta + i\sin\theta$, then $z^{-n} = \cos n\theta - i\sin n\theta$, where $\theta \in \mathbb{R}$ and $n \in \mathbb{Z}^+$.

Express each of the complex numbers $-1 + i\sqrt{3}$ and $\sqrt{3} + i$ in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$.

Let $m, n \in \mathbb{Z}^+$. Show that if $\frac{(-1+i\sqrt{3})^n}{(\sqrt{3}+i)^m} = 8$, then $n = m + 3$ and $n = 4k-1$, where $k \in \mathbb{Z}$.

14. (*a*) Let $f(x) = \dfrac{(x+1)}{(x+2)^2}$ for $x \neq -2$. Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = \dfrac{-x}{(x+2)^3}$ for $x \neq -2$.

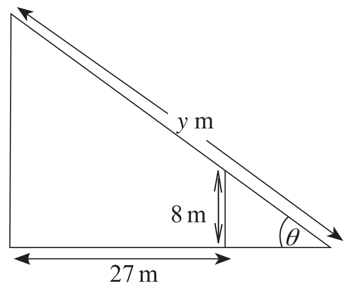
It is given that $f''(x) = \dfrac{2(x-1)}{(x+2)^4}$ for $x \neq -2$, where $f''(x)$ denotes the second derivative of $f(x)$.

Sketch the graph of $y = f(x)$ indicating the asymptotes, turning point and the point of inflection.

(*b*) A fence, 8 m tall, is at a distance of 27 m from a vertical wall of a building. A ladder with its lower end on the horizontal ground goes just over the fence and reaches the wall as shown in the figure. Let y m be the length of the ladder and θ be the angle it makes with the horizontal. Express y as a function of θ.

Show that $\dfrac{dy}{d\theta} = 0$ if and only if $\theta = \tan^{-1}\left(\dfrac{2}{3}\right)$.

By considering the sign of $\dfrac{dy}{d\theta}$ in appropriate intervals, find the length of the shortest such ladder.



15. (*a*) Express $\dfrac{4}{(x-1)(x+1)^2}$ in partial fractions.

Hence, find $\displaystyle\int \frac{1}{(1-e^{-x})(1+e^{x})^2}\,dx$.

(*b*) Using integration by parts, find $\displaystyle\int x^2(\sin x + 2\cos x)\,dx$.

(*c*) Establish the formula $\displaystyle\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx$.

Hence, show that $\displaystyle\int_0^{\pi} \frac{x \sin x}{(2-\sin^2 x)}\,dx = \frac{\pi^2}{4}$.

16. Let $A \equiv (-1, 1)$ and l be the straight line given by $x + y = 7$. Find the coordinates of the points B and C on l such that $A\hat{B}C = A\hat{C}B = \tan^{-1}(7)$.

Also, find the equation of the bisector m of the angle $B\hat{A}C$.

Write down the equation of the circle with BC as a diameter and **hence** write down the equation of any circle through B and C in terms of a parameter.

Deduce the equation of the circle S that passes through the points A, B and C.

Also, find the coordinates of the points of intersection, of the circle S and the straight line m.

17.(*a*) Show that $\cos^3 x \cos 3x + \sin^3 x \sin 3x = \cos^3 2x$.

Hence, solve $8(\cos^3 x \cos 3x + \sin^3 x \sin 3x) = 1$.

(*b*) Let ABC be a triangle. The points D and E are taken on BC such that $BD : DE : EC = 1 : 2 : 3$. Also, let $B\hat{A}D = \alpha$, $D\hat{A}E = \beta$ and $E\hat{A}C = \gamma$. Using the sine rule for suitable triangles, show that $\sin(\alpha+\beta)\sin(\beta+\gamma) = 5\sin\alpha\,\sin\gamma$.

(*c*) Let $|x| \leq 1$, $|y| \leq 1$ and $|z| \leq 1$. Show that if $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \pi$, then

$x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz.$

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(10) Combined Mathematics

Paper II

Part A

1. Two particles of masses m and λm move towards each other on a smooth horizontal table with speeds u and $\frac{2u}{3}$ respectively. Given that the particles move with equal speeds $\frac{u}{2}$ away from each other after their direct impact, show that the coefficient of restitution is $\frac{3}{5}$, and that the value of λ is $\frac{9}{7}$.

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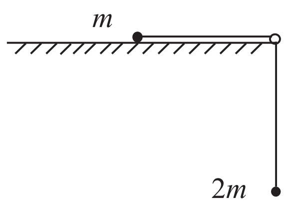
2. A particle of mass m lying on a rough horizontal table is connected by a light inextensible string which is perpendicular to the edge of the table and which passes over a small smooth pulley fixed at the edge of the table, to a particle of mass $2m$ which hangs freely as shown in the figure. The system is released from rest with the string taut. The coefficient of friction between the particle of mass m and the table is $\frac{1}{4}$. Show that the tension in the string is $\frac{5}{6}mg$.



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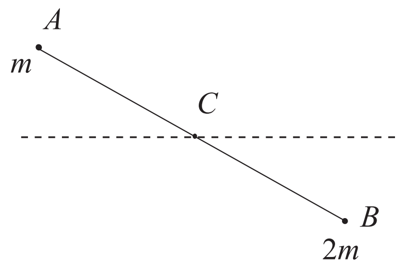
3. Two particles of masses m and $2m$ are attached to the two ends A and B respectively of a light rod AB of length $2a$. The rod is held in a horizontal position with its middle point C smoothly hinged to a fixed point, and released from rest. (See the figure.) Using the Principle of Conservation of Energy, show that the speed v of the particles when the rod makes an angle θ with the horizontal is given by $v^2 = \frac{2ga}{3}\sin\theta$.



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4. Two cars A and B move along a straight road in parallel lanes in the same direction. At time $t = 0$, the cars A and B pass a bridge with speeds u and $\frac{u}{4}$, respectively. Car A moves with the same constant speed u and car B moves with constant acceleration until it reaches the speed $\frac{5u}{4}$ at time $t = T$, and maintains that speed afterwards. In the same diagram, sketch the velocity - time graphs for the motions of car A and car B.

Hence, obtain an equation to determine the time taken by B to overtake A.

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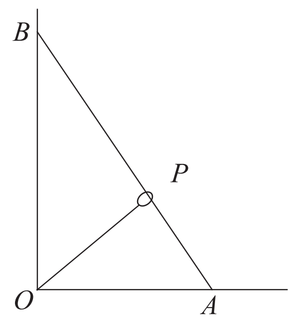
5. A train of mass 300 metric tons is moving in a straight level track, with constant speed $15\,\text{m s}^{-1}$ and the resistance to motion is $50\,\text{N}$ per metric ton. Find the power of the train in kilowatts. Rear coach of mass 50 metric tons gets dislodged while the tractive force of the engine is unaltered. Find the acceleration of the remaining portion of the train.

6. In the usual notation, let $4\mathbf{i} + \mathbf{j}$, $\lambda\mathbf{i} + \mu\mathbf{j}$ and $\mathbf{i} + 5\mathbf{j}$ be the position vectors of three points A, B and C respectively, with respect to a fixed origin O, where λ and μ are positive constants. The diagonals of the quadrilateral $OABC$ are equal in length and perpendicular to each other. Write down $\overrightarrow{AC}$ in terms of $\mathbf{i}$ and $\mathbf{j}$. Using scalar product, show that $\lambda = 4$ and $\mu = 3$.

7. A smooth uniform rod AB of length $2a$ and weight W which passes through a small smooth ring P has its end A on a smooth horizontal ground and the other end B in contact with a smooth vertical wall. The rod is kept in equilibrium, at an angle $60°$ to the horizontal, in a vertical plane perpendicular to the wall by a light inextensible string which connects the ring to the point O as shown in the diagram. Show that $O\hat{P}A = 90°$ and write down equations sufficient to determine the tension of the string.



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8. A particle of mass m is placed on a rough plane inclined at an angle α to the horizontal, where the coefficient of friction between the plane and the particle is μ (< tan α). The particle is held in equilibrium with a force P applied upwards to the particle along a line of greatest slope of the plane. Show that $mg(\sin\alpha - \mu\cos\alpha) \le P \le mg(\sin\alpha + \mu\cos\alpha)$.

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9. Find the probability that the sum of the number of dots obtained in at most three tosses of an unbiased standard die with 1, 2, 3, 4, 5 and 6 dots marked on its six faces, is exactly six.

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10. The mean and the mode of the seven numbers a, b, 4, 5, 7, 4 and 5 are equal, where a and b are positive integers. Find the values of a and b, and show that the variance of the seven numbers is $\frac{6}{7}$.

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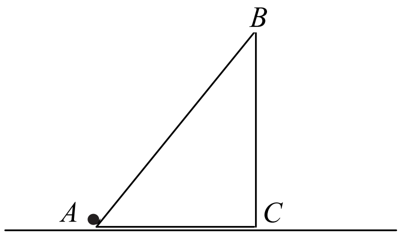
Part B

11. (*a*) A particle projected from a point O on a horizontal ground with a velocity $u = \sqrt{2ga}$ making an angle $\theta \left(0 < \theta < \dfrac{\pi}{2}\right)$ to the horizontal, moves under gravity and hits a target at a point P. The horizontal and vertical distances of P measured from O are a and ka, respectively, where k is a constant. Show that $\tan^2\theta - 4\tan\theta + 4k + 1 = 0$ and **deduce** that $k \le \dfrac{3}{4}$. Now, let $k = \dfrac{11}{16}$. Show that the angle between the two possible directions of projection is $\tan^{-1}\left(\dfrac{4}{19}\right)$.

(*b*) An airport A is situated at an angle θ East of South, at a distance d from an airport B. On a certain day, an air plane flies directly from A to B with a speed u relative to the wind which blows from North at a velocity v ($< u$). Sketch the velocity triangle for this flight, and show that the time taken to fly from A to B is $\dfrac{d}{\sqrt{u^2 - v^2\sin^2\theta} - v\cos\theta}$.

A few days later, the plane flies back directly from B to A with a speed $\dfrac{u}{2}$ relative to the wind which blows from South at a velocity $\dfrac{v}{2}$. Sketch the velocity triangle for the return journey and show that the time taken to fly from B to A is twice as much as the time taken from A to B.

12. (*a*) The triangle ABC in the given figure represents a vertical cross section through the centre gravity of a uniform smooth wedge of mass $3m$. The line AB is a line of greatest slope of the face containing it. Also $B\hat{A}C = \dfrac{\pi}{3}$. The wedge is placed with the face containing AC on a smooth horizontal floor. A particle of mass m is placed at the point A and given a velocity u along $\overrightarrow{AB}$. Assuming that AB is smooth and the particle does not leave the wedge, find the time taken by the particle to come to **rest relative to the wedge.** Now, suppose that the particle gets glued to the wedge in this position. Find also the time taken by the wedge with the particle glued to move an additional distance d.



(*b*) A bead P of mass m is free to move along a smooth circular wire of radius a and centre O which is fixed in a vertical plane. The bead is held at the upper-most point A of the wire and released from rest at a slightly displaced position. Show that the speed v of the bead when OP has turned through an angle θ is given by $v^2 = 2ga(1 - \cos\theta)$.
Find the speed of the bead when the bead reaches the lowest point B. As P reaches the point B, it collides and coalesces with another bead of mass m which is at rest at B. Show that the composite bead Q comes to instantaneous rest when OQ has turned through an angle $\dfrac{\pi}{3}$.

13. One end of a light elastic string of natural length a and modulus mg is attached to a fixed point O. Two equal particles, each of mass m are fastened together to the other end P of the string and the system hangs in equilibrium. Show that the extension of the string in this position is $2a$. Now, one of the particles gets gently detached and the remaining particle of mass m, still attached to the end of the string begins to move. Obtain the equation for motion of P, $\ddot{x}+\frac{g}{a}(x-2a)=0$, where $x(\geq a)$ is the length of the string.

Find the centre C and the amplitude of this simple harmonic motion.

At the point C, a vertical impulse is applied to the particle so that its velocity is trebled. Show that the centre of the motion, while the string is taut remains the same, and that the amplitude of this motion is $3a$.

Hence show that the string becomes slack after a total time $\sqrt{\frac{a}{g}}\left(\frac{\pi}{2}+\sin^{-1}\left(\frac{1}{3}\right)\right)$.

Find the speed of the particle at the instant when the string becomes slack.

14. (*a*) Let $PQRS$ be a parallelogram and let T be the point on QR such that $QT:TR=2:1$. Also, let $\overrightarrow{PQ}=\mathbf{a}$ and $\overrightarrow{PS}=\mathbf{b}$. Express vectors $\overrightarrow{PR}$ and $\overrightarrow{ST}$ in terms of **a** and **b**.

Let U be the point of intersection of PR and ST. Suppose that $\overrightarrow{PU}=\lambda\overrightarrow{PR}$ and $\overrightarrow{SU}=\mu\overrightarrow{ST}$, where λ and μ are scalar constants. By considering the triangle PSU, show that $(\lambda-\mu)\,\mathbf{a}+\left(\lambda+\frac{\mu}{3}-1\right)\mathbf{b}=\mathbf{0}$ and find the values of λ and μ.

(*b*) A system consisting of three forces in the Oxy-plane act at the points indicated below:

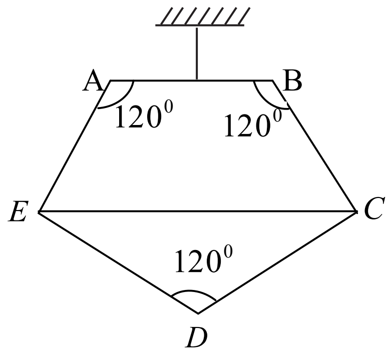
Point	**Position Vector**	**Force**
A	$2a\mathbf{i}+5a\mathbf{j}$	$F\mathbf{i}+3F\mathbf{j}$
B	$4a\mathbf{j}$	$-2F\mathbf{i}-F\mathbf{j}$
C	$-a\mathbf{i}+a\mathbf{j}$	$F\mathbf{i}-2F\mathbf{j}$

Here $\mathbf{i}$ and $\mathbf{j}$ denote unit vectors in the positive directions of coordinate axes Ox and Oy, respectively, and F, a are positive quantities measured in newtons and metres, respectively. Mark these forces in a single diagram, and show that their vector sum is zero.

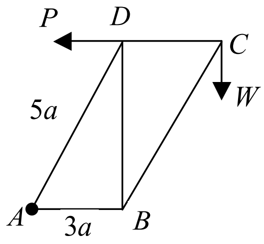
Find the anti-clockwise moment, G, of the system about a point P with position vector $x\mathbf{i}+y\mathbf{j}$ and show that G is independent of x and y. Hence show that the system is equivalent to a couple, and find the moment of this couple.

An additional force $X\mathbf{i}+Y\mathbf{j}$ is now applied at the point D with position vector $\mathbf{d}=-\frac{5a}{2}\mathbf{i}$, so that the resultant of the four forces acting at A, B, C and D, passes through the origin O. Find the values of X and Y.

15. (*a*) The figure represents a frame in the form of a pentagon *ABCDE* formed of five uniform rods of weight *w* **per unit length** jointed at their ends. $AE = BC = 2a$, and $ED = CD = 2b$. The angles at vertices *A, B* and *D* are 120° each. The frame is suspended from the mid-point of *AB* and is in equilibrium with the symmetrical shape maintained by a light rod *CE* of length $2b\sqrt{3}$ connecting the joints *C* and *E*. Show that the reaction at the joint *D* is of magnitude $b\sqrt{3}w$ and find the thrust in the light rod *CE*.

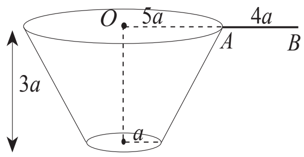


(*b*) The figure represents a framework of light rods *AB*, *BC*, *CD*, *DA* and *DB* freely jointed at their ends, and movable in a vertical plane about the joint *A*. Here $AB = CD = 3a$, $BC = DA = 5a$ and $DB = 4a$. It carries a weight *W* at the joint *C* and equilibrium is maintained with *AB* and *DC* horizontal and *BD* vertical by a horizontal force *P* applied along *CD* at the joint *D*. Find *P* in terms of *W*.
Sketch a stress diagram using Bow's notation and hence find the stresses in all the rods. State whether these are tensions or thrusts.



16. Show, by integration, that the distance of the centre of gravity of a frustum of a uniform hollow right circular cone with circular rims having radii r and $\lambda r(\lambda > 1)$ at a distance h apart, from the centre of its smaller rim is $\frac{h}{3}\left(\frac{2\lambda+1}{\lambda+1}\right)$.

A saucepan is made by fastening the edge of a thin uniform circular plate of radius *a* having a surface density σ to the smaller circular rim of a frustum of height 3*a* of a hollow right circular cone with circular rims of radii *a* and 5*a* having the same surface density σ, and fastening a thin uniform rod *AB* of length 4*a* and linear density ρ to the larger rim of the frustum such that the points *O, A* and are collinear, where *O* is the centre of the larger rim of the frustum, as shown in the figure. Find the position of the centre of gravity of the saucepan.



Show that if $\frac{\rho}{\sigma} < \frac{31}{24}\pi a$, then the saucepan can stay is equilibrium when placed on a horizontal table with its bottom touching it.

It is given that $\rho = \pi a \sigma$. Find the angle *BA* makes with the downward vertical when the saucepan is suspended freely from the end *B*.

17. (*a*) A box contains six red balls, three green balls and three blue balls, which are identical except for colour. A ball is drawn at random from the box. Find the probability that the ball is blue.

If the ball drawn is either green or red, one additional red ball and one additional blue ball are added to the box, together with the original ball. If the ball drawn is blue, there is no replacement.

Now, a second ball is drawn from the box at random. What is the probability that the second ball drawn is blue?
Find the probability that the first ball drawn is a blue one, given that the second ball drawn is a blue one.

(*b*) Marks obtained by 100 students in an examination are given in the following table.

Marks	5 - 19	20 - 34	35 - 49	50 - 64	65 - 79	80 - 94
Mid point (x_i)	12	27	42	57	72	87
Frequency (f_i)	10	20	30	15	15	10

Using the transformation $y_i = \frac{1}{15}(x_i - 42)$, estimate the mean and the variance of this distribution of marks.

The mean and the standard deviation of marks obtained by another 100 students in the same examination are 40 and 15, respectively. Estimate the mean and the variance of the marks obtained by all 200 students in this examination.

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