(11) Higher Mathematics

Structure of the Question Paper

Paper I - **Time : 03 hours** (In addition, 10 minutes for reading.)

This paper consists of **two** parts.

Part A : Ten questions. **All** questions should be answered. 25 marks for each question - altogether 250 marks.

Part B : Seven questions. **Five** questions should be answered. Each question carries 150 marks - altogether 750 marks.

Total marks for paper I 1000

Paper II - **Time : 03 hours** (In addition, 10 minutes for reading.)

This paper consists of **two** parts.

Part A : Ten questions. **All** questions should be answered. 25 marks for each question - altogether 250 marks.

Part B : Seven questions. **Five** questions should be answered. Each question carries 150 marks - altogether 750 marks.

Total marks for paper II 1000

Calculation of the final mark :

Paper I	=	1000 ÷ 20 = 50
Paper II	=	1000 ÷ 20 = 50
Final mark	=	100

Higher Mathematics
Paper I

Important :

* Answer **all** questions of part **A**.
* Answer **five** questions only of part **B**.

Part A

1. Factorize : $x^3(y-z)+y^3(z-x)+z^3(x-y)$.
Hence show that $(a-b)^3(a+b-2c)+(b-c)^3(b+c-2a)+(c-a)^3(c+a-2b)=0$.

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2. Let $k \in \mathbb{R}$. A relation R on $\mathbb{R}$ is defined by xRy if $x^4-y^4-kx^2+ky^2=0$. Show that R is an equivalence relation on $\mathbb{R}$.

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3. Let $f(x) = \dfrac{x+1}{x-1}$ for $x \neq 1$ and let $g(x) = ax^3 + 1$ for $x \in \mathbb{R}$, where a is a real constant. Also let $h(x) = (g \circ f)(x)$ for $x \neq 1$. It is given that $h(2) = 28$. Show that $a = 1$. Write down $h^{-1}(x)$.

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4. Show that $x + y + z$ is a factor of the determinant

$$\Delta = \begin{vmatrix} x & x^3 & y+z \\ y & y^3 & z+x \\ z & z^3 & x+y \end{vmatrix}$$

and express Δ as a product of linear factors.

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5. Find the equation of the normal to the rectangular hyperbola $xy = c^2$ at the point $(ct, \frac{c}{t})$ and show that if it passes through $(0, c)$ then $t^4 + t - 1 = 0$.

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6. Let $a, b \in \mathbb{R}$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} \dfrac{\sqrt[3]{1+ax} - 1}{x} & , \text{ if } x > 0 \\ b & , \text{ if } x = 0 \\ \dfrac{1}{3(1 - e^{\frac{1}{x}})} & , \text{ if } x < 0 \end{cases}$$

If f is continuous at $x = 0$, find the values of a and b.

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7. Let $f\ \mathbb{R} \longrightarrow \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} |x^2 - 1| & \text{if } x \geq -1 \\ -(x^2 - 1) & \text{if } x < -1 \end{cases}$$

Show that f is **not** differentiable at $x = 1$.

Write down $f'(x)$ for all $x \neq 1$.

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8. Using the substitution $z = \dfrac{1}{y}$, transform the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}x} - y \tan x = y^2 \cos^2 x$ into linear form and **hence**, solve it.

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9. Let f and g be real-valued functions defined on the interval $[0,1]$. Suppose that f and g', the derivative of g, are continuous on $[0, 1]$ and that $3f(1-x) + 2xg'(x) = 4x^3$ for $x \in [0, 1]$.

Show that if $\int_0^1 f(x) = 2$ and $g(1) = 1$, then $\int_0^1 g(x)\mathrm{d}x = \frac{7}{2}$

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10. Sketch the curves whose polar equations are given by $r - 2\sin\theta = 0$ and $r^2 - 2r(\sqrt{2}\cos\theta + \sin\theta) + 2 = 0$ in the same diagram.

Show that these curves intersect at right angles.

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Part B

11. (*a*) Let A, B and C be subsets of a universal set S. Stating clearly the laws of Algebra of sets that you use, prove that

(i) $A \cup B = A \cup (A' \cap B)$,

(ii) $B = (A \cap B) \cup (A' \cap B)$ and

(iii) $(A - B) \cap C = (A \cap C) - (B \cap C)$,

where the $A - B$ is defined by $A - B = A \cap B'$.

(*b*) A survey of 150 students was conducted to determine which sports they like from among cricket, hockey and football. It was revealed that 60 students like cricket, 50 like hockey, 70 like football, 35 like hockey and football, 20 like cricket and football, 42 like cricket and hockey and 10 like all three sports. Find the number of students who

(i) do not like any of these three sports,

(ii) like only cricket,

(iii) like at most one sport.

12. (*a*) Let a, b and c be positive numbers.

Show that $\sqrt{a\,b} \leq \frac{1}{2}(a + b)$.

Deduce that $(abc)^{\frac{1}{3}} \leq \frac{1}{3}(a + b + c)$.

Show each of the following :

(i) $(a + 4b)(b + 4c)(c + 4a) \geq 64abc$.

(ii) $a(1 - a)^2 \leq \frac{4}{27}$, for $0 < a < 1$.

(*b*) The transformation $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ in the xy–plane maps the point $(a, a + 2)$ into the point $(2a, b)$, where a and b are real constants. Find the values of a and b. Find the vertices of the parallelogram in the $x'y'$ –plane to which the square in the xy –plane with vertices (0,0), (1,0), (1,1) and (0,1) gets mapped onto.

13. State and prove de Moivre's theorem for a positive integral index.

Let $\omega_k = \cos\left(\frac{2k\pi}{7}\right) + i\sin\left(\frac{2k\pi}{7}\right)$ for $k = 1, 2, 3, \ldots$. Show that $\omega_k^7 = 1$ for $k = 1, 2, 3, \ldots$ and **hence**, write down the six distinct non real roots of the equation $z^7 = 1$.

Show that $1 + \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6 = 0$ and that $\omega_k + \omega_{7-k} = 2\cos\left(\frac{2k\pi}{7}\right)$ for $k = 1, 2, 3$.

Deduce that $\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}$. Also, show that

$$1 + z + z^2 + z^3 + z^4 + z^5 + z^6 = \{z^2 - 2\cos\left(\frac{2\pi}{7}\right)z + 1\}\{z^2 - 2\cos\left(\frac{4\pi}{7}\right)z + 1\}\{z^2 - 2\cos\left(\frac{6\pi}{7}\right)z + 1\}.$$

14. (*a*) Solve the differential equation $(1 - x^2)\frac{dy}{dx} + y = x^2 (1 + x)(1 - x)^{\frac{3}{2}}$ for $-1 < x < 1$ and **hence**, find the solution satisfying $y = 1$ when $x = 0$.

(*b*) Find the differential equation satisfied by the family of curves $y = \lambda(x - 1)^2 + 3$, where λ is a real parameter.

Hence, find the general equation of the family of the orthogonal trajectories.

15. (*a*) Let $I_n = \int_0^1 x^n \cos\left(\frac{\pi}{2} x\right) dx$, where n is a non-negative integer.

Show that $I_n + \frac{8}{\pi^3} n(n-1) I_{n-1} = \frac{2}{\pi}$ for $n \geq 2$.

Hence, find I_4.

(*b*) Let $y = e^{\tan^{-1}x}$. Show that $(1 + x^2)\frac{d^2y}{dx^2} = (1 - 2x)\frac{dy}{dx}$.

Obtain the Maclaurin expansion of y upto and including the term of x^4.

Hence, find an approximate value of the integral $\int_0^{\frac{1}{2}} e^{\tan^{-1}x} dx$

16. (*a*) Show that the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(a\cos\theta, b\sin\theta)$ is given by $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$. The line $y = x + c$ is a tangent to the ellipse $\frac{x^2}{4} + y^2 = 1$. Show that $c = \pm\sqrt{5}$. Find the coordinates of the points P and Q of contact and show that the chord PQ passes through the origin.

(*b*) Show that the area A enclosed by the parabola $y^2 = 4ax$ and the chord joining the points $(ap^2, 2ap)$ and $(aq^2, 2aq)$ is given by $9A^2 = a^4 (p - q)^4$.

Let $P \equiv \left(\frac{4}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right)$. Show that P lies on the parabola $y^2 = \frac{1}{4\sqrt{5}} x$.

Show that the equation of the normal drawn to the parabola $y^2 = \frac{1}{4\sqrt{5}} x$ at the point P is $\sqrt{5}\, y - 8\sqrt{5}\, x + 33 = 0$.

Find the area bounded by the line PQ, the normal $\sqrt{5}\, y - 8\sqrt{5}\, x + 33 = 0$ and the parabola $y^2 = \frac{1}{4\sqrt{5}} x$.

17. (*a*) Let $A = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$ and let $f(x) = \left(\frac{\text{cosec}x + \sec x}{\tan x + \cot x}\right)^2 - \frac{2}{\text{cosec}^2 x}$ for $x \in A$.

Show that $f(x) = \sin 2x + \cos 2x$ for $x \in A$.

Express $f(x)$ in the form $R \sin(\alpha x + \theta)$ for $\alpha > 0$ where R, α and θ are to be determined.

Sketch the graph of f for $x \in A$.

(*b*) The following table gives the values of $f(x) = \frac{1}{1+x^2}$ correct to two decimal places for values of x between 0 and 1 at intervals of length 0.25 :

x	0	0.25	0.50	0.75	1
$f(x) = \frac{1}{1+x^2}$	1	0.94	0.80	0.64	0.50
$xf(x) = \frac{x}{1+x^2}$	0	0.23	0.40	0.48	0.50

Applying **the Simpson rule**, find an approximation for $\int_0^1 \frac{1+x}{1+x^2}\,dx$.

Find the exact value of $\int_0^1 \frac{1+x}{1+x^2}\,dx$

Hence, find an approximation for $\pi + \ln 4$.

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Higher Mathematics
Paper II
Part A

1. The position vectors of three points A, B and C with respect to a fixed origin O are $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, $\beta\mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{i} + 5\mathbf{j} + 5\mathbf{k}$, respectively, where β is a constant. It is given that the point C lies on the plane OAB. Find the value of β.

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2. Two forces $\mathbf{P} = \mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{Q} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ each of magnitude 3 N act at the points A and B with position vectors $3\mathbf{k}$ and $-\mathbf{k}$, respectively. Find their vector sum $\mathbf{R}$ and their moment vector $\mathbf{G}$ about the origin O. **Hence**, show that these two forces reduce to a single resultant force.

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3. A uniform solid right circular cone floats with its vertex above, axis vertical and two thirds of the axis above the free surface in a liquid of constant density ρ. Show that the density of the cone is $\frac{19}{27}\rho$.

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4. The position vector of a particle P at time t, is given by $\mathbf{r} = a(\cos\omega t)\,\mathbf{i} + a(\sin\omega t)\,\mathbf{j} + (c\,\omega t)\,\mathbf{k}$, where a, c and ω are positive constants. Show that the velocity $\mathbf{v}$ of P is of constant magnitude $\omega\sqrt{a^2 + c^2}$ and $\mathbf{v}$ makes a constant angle with the OZ - axis. Find the displacement of P from its initial position, when $t = \frac{2\pi}{\omega}$.

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5. A small smooth sphere moving vertically downwards with speed u strikes a fixed smooth plane of inclination $\frac{\pi}{6}$ to the horizontal and rebounds horizontally. Show that the coefficient of restitution between the sphere and the plane is $\frac{1}{3}$ and that the kinetic energy retained in the sphere is $\frac{1}{3}$ of its value just before impact.

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6. A uniform circular hoop of mass m and radius a can rotate freely in a vertical plane about a horizontal axis through a point A in the hoop. The hoop is held with its centre C vertically above A and then given a small displacement. Show that the velocity of the centre C when it is vertically below A is $\sqrt{2ga}$.

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7. The discrete random variable X takes values ± 3, ± 1 only, with probabilities $P(X = x) = k|x|$, where k is a positive constant.

(i) Find the value of k and $E(X^2)$.

(ii) Show that the standard deviation of X is $\sqrt{7}$.

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8. The random variable X takes values 0, 1, 2 and 3 only. Given $P(X \leq 1) = 0.5$, $P(X \leq 2) = 0.9$ and $E(X) = 1.3$, obtain the probability distribution of X. Show that the value of $\mathrm{Var}(X)$ is approximately equal to 1.

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9. The probability density function $f(x)$ of a continuous random variable X defined only for non-negative x, is as follows : $f(x) = kx$, for $0 \leq x \leq 1$ and $f(x) = \frac{k}{x^4}$, for $x \geq 1$.

Find,

(i) the value of the constant k,

(ii) $E(X)$, the mean of X, and

(iii) the mode of this probability distribution.

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10. The cumulative distribution function $F(x)$, of a random variable X that is defined for $0 \leq x \leq 1$ is given by $F(x) = ax^2 - 2x^3$. Find the value of the constant a, and show that $E(X) = \frac{1}{2}$. Also, find the probability $P(\frac{1}{4} \leq X \leq \frac{3}{4})$.

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Part B

11. A system consists of six forces each of magnitude P N act (in the directions as indicated), along the six edges $\overrightarrow{OA}$, $\overrightarrow{AB}$, $\overrightarrow{OB}$, $\overrightarrow{BC}$, $\overrightarrow{OC}$, $\overrightarrow{CA}$ of a regular tetrahedron $OABC$. Three vertices A, B, C of the tetrahedron have coordinates, $(a, a, 0)$, $(a, 0, a)$ and $(0, a, a)$ where a is a length measured in metres, with respect to Cartesian axes Ox, Oy, Oz, with the vertex O as the origin. Write down unit vectors in the directions of these forces and **hence**, express the six forces in vector form. Show that, the system can be reduced to a force $\underline{R}$ of magnitude $R = \sqrt{6}\,P$ N acting at the origin O and a couple of moment vector $\underline{G}$. Express $\underline{R}$ and $\underline{G}$ in terms of unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ and the scalar P.

Hence show that the system is equivalent to a wrench of pitch $p = \dfrac{\underline{R}\,.\,\underline{G}}{R^2}$, and that its axis is along the line whose vector equation is $\underline{G} - \underline{r} \times \underline{R} = p\underline{R}$. Find p in terms of a and obtain the position vector equation of the axis of the wrench, in the standard parametric form $r = \underline{r}_0 + \lambda\underline{N}$, where suitable vectors $\underline{r}_0$ and $\underline{N}$ are to be found. Deduce Cartesian equation of the axis of the wrench, and the direction cosines of this line.

12. A circular plate of radius a is completely immersed vertically in a homogeneous liquid, of density ρ with its centre O at a depth $h(\geq a)$ below the free surface of the liquid. Write down the liquid thrust on the plate. Using integration, show that the centre of pressure of the plate lies on its vertical diameter at a depth $\dfrac{a^2}{h}$ below the centre O..

A plane door S in the shape of the region between two concentric circles of radii a and $2a$ is located on a vertical side of a tank filled with a homogeneous liquid of density ρ. The depth of the liquid in the tank is $6a$. The door is freely hinged at its uppermost point A which is at a depth a below the free surface. Find the force that should be applied perpendicular to the door at its lowest point B to keep it closed.

13. A particle is projected vertically upwards with speed U from a point A on the horizontal ground, in a medium which offers a resistance kv per unit mass when its speed is v, where k is a constant. Show that, the particle comes to rest instantaneously after a time $T = \dfrac{1}{k}\ln\left(\dfrac{g + kU}{g}\right)$, at the point B at a height H above A, where $kH = U - gT$.

If the time taken by the particle for its downward motion, starting from rest at B, until it reaches A is T_1 and its speed at A is U_1 show that $T + T_1 = \dfrac{1}{k}\ln\left(\dfrac{g + kU}{g - kU_1}\right)$.

14. A particle P of mass m is attached to one end of a light inextensible string of length $2a$, and another particle Q of equal mass is attached to the other end of the string. Particle P is held at a point A and particle Q held at the point B distant a vertically below the point A. Initially, the particle P is given a horizontal velocity u and simultaneously, particle Q is released from rest at point B.

By considering the motion of particle P relative to Q, or otherwise, show that when the string becomes taut the inclination of the string to the vertical is $\frac{\pi}{3}$.

By considering conservation of angular momentum of the system about its centre of mass, G, show further that

(i) just after the string becomes taut **and also** in the subsequent motion of the system, the angular velocity of the string remains constant and equal to $\frac{u}{4a}$;

(ii) the time at which the string becomes vertical with P below Q is $t_1 = \frac{a}{u}\left(\sqrt{3} + \frac{8\pi}{3}\right)$.

By considering the motion of G, and the motion of the system relative to G

(iii) show that the path of G in the subsequent motion of the system is a parabola, and find the horizontal and vertical distances of G from the point A, at time $t = t_1$.

15. State the moment of inertia of a uniform circular ring of mass M and radius a about an axis through its centre and perpendicular to its plane.

Show, by integration that the moment of inertia of a uniform circular disc of mass M and radius a about an axis through its centre and perpendicular to its plane is $\frac{1}{2}Ma^2$.

The ring and the disc independently roll (without slipping) down lines of greatest slope of a fixed plane inclined at an angle α to the horizontal, each body starting from rest with their centres on the same straight horizontal line, at time $t = 0$.

Using the principle of conservation of energy, show that the speeds v and V, of the centres of the ring and the disc, after rolling through a distance x down the plane are given by $v^2 = gx \sin \alpha$ and $V^2 = \frac{4g}{3} x \sin \alpha$, respectively.

Hence, or otherwise, find the acceleration of each body, and show that the disc moves through a distance $\frac{1}{12} gt^2 \sin \alpha$ more than the distance moved by the ring.

16. (*a*) Let X denote the random variable "the number of successes in n independent trials, each with success probability p $(0 < p < 1)$.", and let X follow a binomial distribution, with probability function $P(X = x) = {}^{n}C_{x}\,(1 - p)^{n-x}\,p^{x},\ x = 0, 1, 2, \ldots n.$

Show that $P(X = x) \leq P(X = x + 1)$, if and only if $x \leq (n + 1)\,p - 1$.

Suppose a particular marksman makes several independent trials to hit a specified target, and the probability that he succeeds in hitting the target in each trial is 0.3.

(i) If the number of trials the marksman makes is 8, find the number of successes with the highest probability.

(ii) Find the least number of trials the marksman should make, in order that the probability of hitting the target at least once is greater than 80%.

(*b*) A discrete random variable R follows a geometric distribution whose probability function is given by $P(R = r) = q^{r-1}p,\ r = 1, 2, 3, \ldots$, where $0 < p < 1$ and $q = 1 - p$.

Show that

(i) the cumulative distribution function, $P(R \leq r) = 1 - q^{r}$, and

(ii) $P(R > s + t \mid R > s) = P(R > t)$, for any two positive integers s and t.

17. (*a*) Time interval X (in minutes) between consecutive arrival times of buses (in a certain route) at a bus stop A is a random variable and X follows an exponential distribution with density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here λ is a positive parameter. Find the mean μ and the standard deviation σ of the random variable X, in terms of the parameter λ.

Now let the parameter $\lambda = \frac{1}{10}$ and suppose that a bus (in this particular route) arrived at the stop A at 7.00 *a.m*. Find the probability that the next bus (in the same route) will arrive at the stop A between 7.15 *a.m*. and 7.30 *a.m*.

[**It may be assumed that** $e^{-1.5} \approx 0.2231$.]

(*b*) The times Y (in minutes) taken by an express train to travel from one station S_1 to the next station S_2 is a random variable which follows a normal distribution with a mean of 40 minutes and a standard deviation of 5 minutes.

Suppose an express train left the station S_1 at 2.00 *p.m.* and started to travel towards the station S_2,

(i) Find the probability that the train will reach station S_2 before 2.45 *p.m*.

(ii) Given that the train had arrived at station S_2 before 2.45 *p.m*., find the probability that the train had arrived at station S_2 before 2.30 *p.m*.

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