MATHEMATICS

Grade 11

Part - I

Educational Publications Department

To obtain electronic textbooks, visit
www.edupub.gov.lk

First Print	2015
Second Print	2016
Third Print	2017
Fourth Print	2018
Fifth Print	2019

All Rights Reserved

ISBN 978-955-25-0309-2

Published by Educational Publications Department
Printed by State Printing Corporation
Panaluwa, Padukka.

The National Anthem of Sri Lanka

Sri Lanka Matha
Apa Sri Lanka Namo Namo Namo Namo Matha
Sundara siri barinee, surendi athi sobamana Lanka
Dhanya dhanaya neka mal palaturu piri jaya bhoomiya ramya
Apa hata sepa siri setha sadana jeewanaye matha
Piliganu mena apa bhakthi pooja Namo Namo Matha
Apa Sri Lanka Namo Namo Namo Namo Matha
Oba we apa vidya
Obamaya apa sathya
Oba we apa shakthi
Apa hada thula bhakthi
Oba apa aloke
Apage anuprane
Oba apa jeevana we
Apa mukthiya oba we
Nava jeevana demine, nithina apa pubudukaran matha
Gnana veerya vadawamina regena yanu mana jaya bhoomi kara
Eka mavakage daru kela bevina
Yamu yamu vee nopama
Prema vada sema bheda durerada
Namo, Namo Matha
Apa Sri Lanka Namo Namo Namo Namo Matha

අපි වෙමු එක මවකගෙ දරුවෝ
එක නිවසෙහි වෙසෙනා
එක පාටැති එක රුධිරය වේ
අප කය තුළ දුවනා

එබැවිනි අපි වෙමු සොයුරු සොයුරියෝ
එක ලෙස එහි වැඩෙනා
ජීවත් වන අප මෙම නිවසේ
සොඳින සිටිය යුතු වේ

සැමට ම මෙත් කරුණා ගුණෙනී
වෙළී සමගි දමිනී
රන් මිණි මුතු නො ව එය ම ය සැපතා
කිසි කල නොම දිරනා

ආනන්ද සමරකෝන්

ஒர தாய் மக்கள் நாமாவோம்
ஒன்றே நாம் வாழும் இல்லம்
நன்றே உடலில் ஓடும்
ஒன்றே நம் குருதி நிறம்

அதனால் சகோதரர் நாமாவோம்
ஒன்றாய் வாழும் வளரும் நாம்
நன்றாய் இவ் இல்லினிலே
நலமே வாழ்தல் வேண்டுமன்றோ

யாவரும் அன்பு கருணையுடன்
ஒற்றுமை சிறக்க வாழ்ந்திடுதல்
பொன்னும் மணியும் முத்துமல்ல - அதுவே
யான்று மழியாச் செல்வமன்றோ.

ஆனந்த சமரக்கோன்
கவிதையின் பெயர்ப்பு.

Being innovative, changing with right knowledge
Be a light to the country as well as to the world.

<u>Message from the Hon. Minister of Education</u>

The past two decades have been significant in the world history due to changes that took place in technology. The present students face a lot of new challenges along with the rapid development of Information Technology, communication and other related fields. The manner of career opportunities are liable to change specifically in the near future. In such an environment, with a new technological and intellectual society, thousands of innovative career opportunities would be created. To win those challenges, it is the responsibility of the Sri Lankan government and myself, as the Minister of Education, to empower you all.

This book is a product of free education. Your aim must be to use this book properly and acquire the necessary knowledge out of it. The government in turn is able to provide free textbooks to you, as a result of the commitment and labour of your parents and elders.

Since we have understood that the education is crucial in deciding the future of a country, the government has taken steps to change curriculum to suit the rapid changes of the technological world. Hence, you have to dedicate yourselves to become productive citizens. I believe that the knowledge this book provides will suffice your aim.

It is your duty to give a proper value to the money spent by the government on your education. Also you should understand that education determines your future. Make sure that you reach the optimum social stratum through education.

I congratulate you to enjoy the benefits of free education and bloom as an honoured citizen who takes the name of Sri Lanka to the world.

Akila Viraj Kariyawasam
Minister of Education

Foreword

The educational objectives of the contemporary world are becoming more complex along with the economic, social, cultural and technological development. The learning and teaching process too is changing in relation to human experiences, technological differences, research and new indices. Therefore, it is required to produce the textbook by including subject related information according to the objectives in the syllabus in order to maintain the teaching process by organizing learning experiences that suit to the learner needs. The textbook is not merely a learning tool for the learner. It is a blessing that contributes to obtain a higher education along with a development of conduct and attitudes, to develop values and to obtain learning experiences.

The government in its realization of the concept of free education has offered you all the textbooks from grades 1-11. I would like to remind you that you should make the maximum use of these textbooks and protect them well. I sincerely hope that this textbook would assist you to obtain the expertise to become a virtuous citizen with a complete personality who would be a valuable asset to the country.

I would like to bestow my sincere thanks on the members of the editorial and writer boards as well as on the staff of the Educational Publications Department who have strived to offer this textbook to you.

W. M. Jayantha Wickramanayaka,
Commissioner General of Educational Publications,
Educational Publications Department,
Isurupaya,
Battaramulla.
10.04.2019

Monitoring and Supervision

W. M. Jayantha Wickramanayaka - Commissioner General of Educational Publications
Educational Publications Department

Direction

W.A. Nirmala Piyaseeli - Commissioner of Educational Publications (Development)
Educational Publications Department

Co-ordination

Thanuja Maithrie Witharana - Assistant Commissioner
Educational Publications Department

Chandima de Zoysa - Assistant Commissioner (2019 reprinting)
Educational Publications Department

Board of Editors

Dr. Romaine Jayewardene - Senior Lecturer, University of Colombo
Dr. D.K. Mallawa Arachchi - Senior Lecturer, University of Kelaniya
Dr. Sri Dharan - Senior Lecturer, University of Colombo
Dr. Nalin Ganegoda - Senior Lecturer, University of Sri Jayawardenapura
B.D.C Biyanwila - Director of Education, Ministry of Education
G.P.H. Jagath Kumara - Senior Lecturer, National Institute of Education, Maharagama
Thanuja Maithrie Witharana - Assistant Commissioner
Educational Publications Department

Board of Writers

Dr. Rochana Meegaskumbura - Senior Lecturer, University of Peradeniya
Dr. J. K. Rathnayake - Senior Lecturer, University of Colombo
Dr. Jayantha Senadeera - Senior Lecturer, Sri Lanka Open University
H.M.A. Jayasena - In Service Adviser (Retired)
Y.V.R. Witharama - In Service Adviser, Zonal Education Office, Dehiowita
W.M.W.C. Walisinghe - In Service Adviser, Zonal Education Office, Kegalle
Ajith Ranasinghe - In Service Adviser, Zonal Education Office, Homagama
Anura Weerasinghe - In Service Adviser, (Priven), Matara District
W.M.D. Lal Wijekantha - Teacher, St. Thomas College, Mt. Lavinia
N. Vakeeshamoorthi - Director of Education (Retired)
R.S.E. Pushparajan - Assistant Director, Zonal Education Office, Puttalam
V. Murali - Teacher Educator Service, Zonal Education Office, Vavuniya

Language Editing

M.A.K. Samurdhi - Teacher, Sirimavo Bandaranaike Vidyalaya, Colombo 07

Proof Reading

Edward R. Peiris - Engineer (Retired)

Illustrations, Cover Page and Technical Assistance

B.T. Chathurani Perera - Educational Publications Department
R.D. Thilini Sewwandi - Educational Publications Department

Message of the Board of Compilers

This textbook has been compiled in accordance with the new syllabus to be implemented from 2015.

Textbooks are compiled for students. Therefore, we have made an attempt to compile this textbook in a simple and detailed manner making it possible for you to read and understand it on your own.

We have included descriptions, activities and examples to introduce the subject concepts in an attractive manner and to establish them. Moreover, activities are organized from simple to complex to develop an interest to do them.

We have used the terms related to mathematical concepts in accordance with the glossary of technical terms of mathematics compiled by the Department of Official Languages.

Some subject matter learnt during the earlier grades is necessary to learn the subject content in the grade 11 syllabus. Thus, review exercises are included at the beginning of each chapter to revise previous knowledge. You will be prepared by them for the subject content of grade 11.

In addition, students may use the grade 10 book which you have if you need to recall previuos knoweledge.

You will gain maximum benefit from this textbook by reading the chapters and doing the review exercises of each chapter even before your teacher teaches them in the classroom.

We hope that studying mathematics will be an interesting, joyful and productive experience.

Board of Compilers

Contents

පිවිසෙන්න නිදහසේ සතුටින් දැනුම සොයා..
The National e-learning Portal for the General Education
ට... www.e-thaksalawa.moe.gov.lk

1 Real Numbers

By studying this lesson will be able to

- to investigate number sets
- workout basic mathematical operations regarding surds.

It is believed that the concept of numbers originated among the human race about 30 000 years ago. This concept which originated and developed independently in various civilizations, evolved globally and has now become a universal field of study named mathematics.

It can be assumed that numbers were initially used in early civilizations for simple purposes such as counting and accounting. There is no doubt that the first numerical concepts that were developed were "one" and "two". Later the concepts of three, four etc., must have been developed. Then man would have realized that he could name any amount that he wished in this manner. Different civilizations used different symbols to name numbers.

It is accepted based on historical evidence, that the numerals 1, 2, 3 etc., which we now use, originated in India. The honour of being the first to use the concept of zero as a number as well as being the first to introduce the positional decimal number system also goes to India. This number system is now defined as the Hindu-Arabic number system and the modern belief is that it was first taken to the Middle-East and then to Europe by traders. This system is the standard number system which is accepted and used worldwide now.

The manipulation of numbers using the basic mathematical operations (addition, subtraction, multiplication and division) can be considered as a great revolution in the history of mankind in relation to the use of numbers. In this age of technology it is unimaginable to think of the existence of man without numbers and the operations performed on them.

Although the numbers 1, 2, 3 etc., can be considered as the first numbers that were used to fulfill certain needs of man, later the number zero, fractional numbers and negative numbers were also included in the number system. During the period when mathematics was developing as a separate field, the attention of mathematicians was directed towards various other types of numbers (sets) too. In this lesson we hope to study about such sets of numbers, their notations and properties.

The Set of Integers ($\mathbb{Z}$)

It is natural that we identify initially the numbers 1, 2, 3, … which we first learnt about as children. These numbers are defined as **counting numbers** and the set which consists of all these numbers is written using set notation as follows.

$$\{1, 2, 3, \ldots\}$$

The reason for this set of numbers to be called the counting numbers is very clear. However its mathematical usage in modern times is limited. The name used most often now for this set is "**the set of positive integers**". This set is denoted by $\mathbb{Z}^+$.

$$\text{Thus, } \mathbb{Z}^+ = \{1, 2, 3, \ldots\}.$$

That is, the numbers 1, 2, 3, … are called positive integers.

The numbers defined as negative integers are -1, -2, -3, etc. Although there is no commonly used symbol to denote this set, some mathematicians, based on the needs of their field of study, use the symbol $\mathbb{Z}^-$.

The positive integers, zero and the negative integers together form the set of **integers**. This set is denoted by $\mathbb{Z}$. Accordingly,

$$\mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

or equivalently,

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}.$$

The Set of Natural Numbers ($\mathbb{N}$)

Let us consider again the set of numbers 1, 2, 3,... This set is also defined as the set of **natural numbers** and is denoted by $\mathbb{N}$.

$$\text{That is, } \mathbb{N} = \{1, 2, \ldots\}.$$

Note: There is no consensus among mathematicians regarding which numbers should be considered as natural numbers. The suitability of calling the numbers 1, 2, 3, … natural numbers is clear. However some of the mathematicians (especially specialists in set theory), have considered 0 as a natural number in their books.

One reason may be because at that time there was no accepted name nor accepted symbol for the set consisting of 0 and the positive integers.However most books on number theory consider the set of natural numbers to be the set $\{1, 2, 3, ...\}$.Almost all authors of mathematics books now mention at the beginning of their books which set of numbers they consider as the natural numbers.

The set of Rational Numbers ($\mathbb{Q}$)

We have come across earlier that, like the integers, fractions too can be considered as numbers, and that operations such as addition and multiplication can be performed on them too. Every integer can be written as a fraction. (For example, we can write $2 = \frac{2}{1}$). Further, a fraction can be written in different forms, all having the same numerical value. (For example, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$). We have also come across negative fractions ($-\frac{2}{5}, -\frac{11}{3}$, etc.). Although we usually think that the numerator and denominator of a fraction should consist of integers, this is actually not the case. For example, $\frac{3}{\sqrt{2}}$ is also a fraction. However, fractions with integers in both the numerator and the denominator (apart from 0 in the denominator), have an important place in mathematics. They are called **rational numbers**. The set of rational numbers is denoted by $\mathbb{Q}$. Accordingly, the set of rational numbers can be defined using the set builder notation as follows.

$$\mathbb{Q} = \left\{ \frac{a}{b} \ : a, b \in \mathbb{Z} \text{ and } b \neq 0 \right\}.$$

There are other ways too of defining the set of rational numbers. One other way is as follows.

$$\mathbb{Q} = \left\{ \frac{a}{b} \ : a \in \mathbb{Z}, b \in \mathbb{Z}^+ \right\}.$$

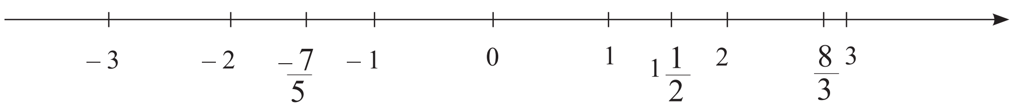
Both these definitions are equivalent. Since the denominator of a rational number cannot be 0 and since all the negative rational numbers can be obtained by considering the fractions with the negative integers in the numerator and positive integers in the denominator.

The Set of Irrational Numbers ($\mathbb{Q}'$)

It is appropriate to define the irrational numbers now. Do you recall how you learnt about numbers in previous grades by drawing a number line? Let us reconsider this now.

Let us consider a straight line which can be lengthened as required in either direction. Let us name a point we like on that line as the origin 0. Let us assume that we have marked all the numbers 1, 2, 3, etc., on one side of 0 (usually the right hand side) and all the numbers – 1, – 2, – 3 etc., on the opposite side, keeping equal gaps between the numbers. That is, let us assume that the points corresponding to all the integers have been marked on this number line. Let us also assume that the points corresponding to all the rational numbers too have been marked on this line.

The figure below shows several such points that have been marked.



Accordingly, all the rational numbers (including the integers) are now assumed to have been marked on this line. Now, do you think that corresponding to each point on the line, a number has been marked? If asked differently, do you think that the distance from 0 to each point on the line can be written as a rational number? In truth, there are several points remaining on the number line which have not been marked. That is, there are points remaining on this number line that cannot be represented by a rational number. It is clear that the points that are remaining are those which correspond to the numbers which **cannot** be written in the form $\frac{a}{b}$, where a and b are integers. The numbers which correspond to the remaining points are defined as irrational numbers.

There is no specific symbol to denote the set of irrational numbers and it is usually denoted by $\mathbb{Q}'$, the complement of the set $\mathbb{Q}$. The numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ can be given as examples of irrational numbers. In fact, the square root of any positive integer which is not a perfect square is an irrational number. Apart from these, mathematicians have proved that π, which is the ratio of the circumference of any circle to its diameter, is also an irrational number. We take the value of π to be $\frac{22}{7}$ as an approximate value, for the convenience of performing calculations.

The Set of Real Numbers ($\mathbb{R}$)

According to the above discussion, all the numbers on a number line can be represented by rational numbers or irrational numbers. We call all the rational numbers together with all the irrational numbers, that is, all the numbers that can be represented on a number line, the **real numbers**. The set of real numbers is denoted by $\mathbb{R}$.

The Decimal Representation of a Number

Any real number can be represented as a decimal number. Initially, let us consider the decimal representation of several rational numbers.

1. The decimal representation of a rational number

$$4 = 4.000\ ...$$

$$\frac{1}{2} = 0.5 = 0.5000\ ...$$

$$\frac{11}{8} = 1.375 = 1.375000\ ...$$

$$\frac{211}{99} = 2.131313...$$

$$\frac{767}{150} = 5.11333...$$

$$\frac{37}{7} = 5.285714285714285714\ ...$$

A common property of these decimal representations is that starting at a certain point to the right of the decimal point (or from the beginning), one set of numerals (or one numeral) is recurring. Recurring means that it keeps repeating itself.

For example, in the decimal representation of $\frac{1}{2}$, the numeral 0 recurs starting from the second decimal place. The numeral 0 recurs from the first decimal place in the decimal representation of 4, the pair of numerals 13 recurs from the beginning in the decimal representation of $\frac{211}{99}$ and the group of numerals 285714 recurs from the beginning in the decimal representation of $\frac{37}{7}$. This property, that is, a group of numerals recurring continuously, is a property common to all rational numbers. If the portion that recurs is just 0, such a decimal representation is defined as a **finite decimal** (or **terminating decimal**). The decimals of which the portion that recurs is not zero, are called **recurring decimals**. Accordingly, $\frac{1}{2}$, 4 and $\frac{11}{8}$ in the above example are finite decimals while the rest are recurring decimals.

The above discussion leads to the following statement.

Every rational number can be written as a finite decimal or a recurring decimal.

Let us now learn a marvelous result regarding rational numbers. Suppose the rational number $\frac{a}{b}$ has a finite decimal representation. Let us assume that a and b have no common factors. Then the denominator (that is, b) has only powers of 2 or 5 (or both) as its factors. A rational number which has a recurring decimal representation must have a prime factor other than 2 and 5 in its denominator.

Recurring decimals are written in a concise form, by placing a dot above a numeral or numerals as shown in the following examples to indicate that they are recurring.

Recurring Decimal	Written Concisely
12.4444	$12.\dot{4}$
2.131313...	$2.\dot{1}\dot{3}$
5.11333...	$5.11\dot{3}$
5.285714285714285714...	$5.\dot{2}8571\dot{4}$

Exercise 1.1

1. For each of the following rational numbers state whether it is a finite decimal or a recurring decimal. Express the fractions which are recurring decimals in decimal form and then write them in a concise form.

a. $\frac{3}{4}$ **b.** $\frac{5}{5}$ **c.** $\frac{3}{7}$ **d.** $\frac{5}{9}$ **e.** $\frac{5}{21}$ **f.** $\frac{7}{32}$

g. $\frac{19}{33}$ **h.** $\frac{13}{50}$ **i.** $\frac{7}{64}$ **j.** $\frac{5}{18}$ **k.** $\frac{15}{128}$ **l.** $\frac{41}{360}$

2. The decimal representation of an irrational number

Finally, let us consider the decimal representation of an irrational number. In the decimal representation of an irrational number, no group of numerals recurs. For example, when the decimal representation of $\sqrt{2}$ is written down up to 60 decimal places, we obtain the following:

1.414213562373095048801688724209698078569671875376948073176679

π, which is a number we come across often is also an irrational number. When the value of π is calculated up to 60 decimal places we obtain the following:

3.141592653589793238462643383279502884197169399375105820974944

The following statements can be made regarding irrational numbers.

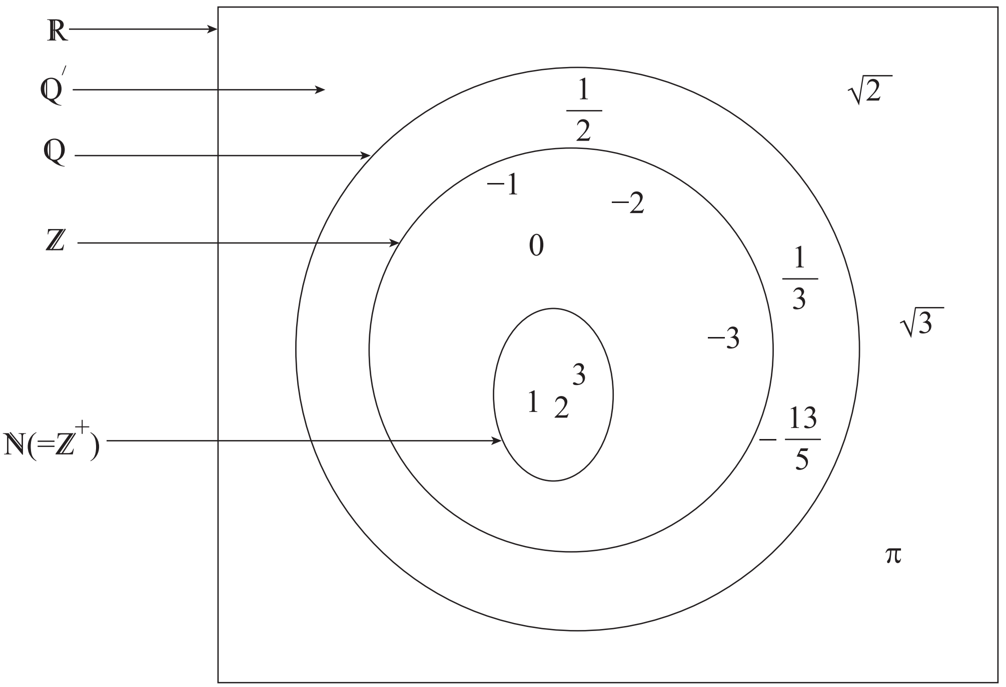
There is no group of recurring numerals in the decimal representation of an irrational number. If the decimal representation of a number is not finite, it is called **an infinite decimal**. Accordingly rational numbers which are recurring decimals and irrational numbers have infinite decimal representations.

Note: When describing the decimal representation of irrational numbers, one frequent error that is made is stating that "there is no pattern in the decimal representation of an irrational number". The issue here is that the word "pattern" is not well defined in mathematics. For example, the following decimal number has a very clear pattern.

0.101001000100001000001...

However, this is an irrational number, Observe that there is no group of recurring numbers.

All the sets of numbers that have been studied so far can be represented in a Venn diagram as follows, with the set of real numbers as the universal set, and the other sets as its subsets. A few numbers belonging to each subset have been included to make it easier for you to understand the relationship between the sets.



Exercise 1.2

1. State whether the following real numbers as rational numbers or irrational numbers.

a. $\sqrt{2}$ **b.** $\sqrt{25}$ **c.** $\sqrt{6}$ **d.** $\sqrt{11}$ **e.** 6.52

2. Determine whether each of the following statement is true or false.

a. Any real number is a finite decimal or an infinite decimal.
b. There can be rational numbers with infinite decimals representations.
c. Any real number is a recurring decimal or an infinite decimal.
d. 0.010110111011110... is a rational number.

1.2 Surds

There is no doubt that you recall how numerical (and algebraic) expressions are written using the symbol " $\sqrt{}$ " which is defined as the radical sign. For example, $\sqrt{4}$ is defined as the **positive square root** of 4 and it represents the positive number which when squared is equal to 4; that is 2. The positive square root is referred to as square root too. If a certain positive integer x is such that its square root $\sqrt{x}$ is also a positive integer, then x is defined as a perfect square. Accordingly, 4 is a perfect square. Since $\sqrt{4}$ is equal to 2. However, $\sqrt{2}$ is not the square root of a perfect square. We observed earlier that $\sqrt{2}$ is approximately equal to 1.414. We also learnt earlier in this lesson that $\sqrt{2}$ is an irrational number. A numerical term involving the symbol $\sqrt{}$ of which the value is not a rational number is defined as a surd.

The radical sign "$\sqrt{}$ " is used not only to denote the square roots of numbers, but also other roots. For example, $\sqrt[3]{2}$ denotes the positive number which when raised to the power 3 is equal to 2. This is called the cube root of 2. This is also an irrational number. Its value is approximately 1.2599. (You can verify this by finding the value of 1.2599^3). We can define the fourth root of 2 and the fifth root of 2 etc., in a similar manner. Such definitions can be made for other positive numbers as well; for example, $\sqrt[3]{5}$ and $\sqrt[6]{8.24}$. Such expressions are also surds. However, in this lesson we will only consider surds which are square roots of positive integers.

The square root of a positive integer which is not a perfect square is neither a finite decimal nor a recurring decimal. Observe that surds are therefore always irrational numbers.

Here our focus is on simplifying expressions which involve surds. There are many reasons why such simplifications are important. One reason is that it facilitates calculations. For example, when it is necessary to find the value of $\frac{1}{\sqrt{2}}$, if we take $\sqrt{2}$ to be approximately equal to 1.414, we would need to find the value of $\frac{1}{1.414}$. This division is fairly long. However, by simplifying this in the following manner, calculations are made easier.

$$\frac{1}{\sqrt{2}} = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} \quad \text{(Multiplying the numerator and denominator by } \sqrt{2}\text{)}$$

$$= \frac{\sqrt{2}}{2}$$

$$= \frac{1.414}{2} = 0.707.$$

Another reason for simplifying surds is to minimize errors during calculations.

For example, let us find the value of $\frac{\sqrt{20}}{2} - \sqrt{5}$. Let us use 4.5 as an approximate value for $\sqrt{20}$, and 2.2 as an approximate value for $\sqrt{5}$. Then

$$\begin{aligned}\frac{\sqrt{20}}{2} - \sqrt{5} &= \frac{4.5}{2} - 2.2\\ &= 2.25 - 2.2\\ &= 0.05\end{aligned}$$

However, the actual value of this expression is 0. One reason for getting a different answer is because we used approximate values for $\sqrt{20}$ and $\sqrt{5}$. However, by simplifying the above expression in a different way, we can get the correct value which is 0.

Surds appear in various form.

A special feature of a surd of the form $\sqrt{20}$ is that the whole number is under the radical sign. Such surds are defined as **entire surds**. $6\sqrt{15}$ means $6 \times \sqrt{15}$. This is a product of a surd and a rational number (not equal to 1). This is not an entire surd.
A surd is in its simplest form when it is in the form $a\sqrt{b}$; where a is a rational number and b has no factors which are perfect squares. For example $6\sqrt{15}$ is in its simplest form but $5\sqrt{12}$ is not, since 4 which is a perfect square is a factor of 12.

Now let us consider how to simplify expressions that contain surds of various forms.

Example 1

Simplify $3\sqrt{5} + 6\sqrt{5}$.

This can be simplified by considering $\sqrt{5}$ to be an unknown term.

$$3\sqrt{5} + 6\sqrt{5} = 9\sqrt{5}.$$

This simplification is similar to the simplification $3x + 6x = 9x$.

Observe that the expression obtained above in surd form cannot be simplified further. Keep in mind that simplifying further by using an approximate value for $\sqrt{5}$ is not what is meant by simplifying surds.
You should also keep in mind the important fact that an expression of the form $3\sqrt{2} + 8\sqrt{3}$ cannot be simplified further.

Now let us through examples, consider how expressions with surds are simplified by applying the properties of indices.

Example 2

Express the entire surd $\sqrt{20}$ as a surd.

$$\begin{aligned}\sqrt{20} &= \sqrt{4 \times 5} \\ &= \sqrt{4} \times \sqrt{5} \qquad (\text{Since } \sqrt{ab} = \sqrt{a} \times \sqrt{b}) \\ &= 2 \times \sqrt{5} \\ &= \underline{\underline{2\sqrt{5}}}\end{aligned}$$

Example 3

Express the surd $4\sqrt{5}$ as an entire surd.

$$\begin{aligned}4\sqrt{5} &= \sqrt{16} \times \sqrt{5} \qquad (\text{Since } 4 = \sqrt{16}) \\ &= \sqrt{16 \times 5} \\ &= \underline{\underline{\sqrt{80}}}\end{aligned}$$

Next let us consider how multiplication and division are performed on surds.

Example 4

Simplify: $5\sqrt{3} \times 4\sqrt{2}$.

Let us multiply the rational and irrational parts separately.

$$\begin{aligned}5\sqrt{3} \times 4\sqrt{2} &= 5 \times 4 \times \sqrt{3} \times \sqrt{2} \\ &= 20 \times \sqrt{3 \times 2} \\ &= \underline{\underline{20\sqrt{6}}}\end{aligned}$$

Example 5

Simplify: $3\sqrt{20} \div 2\sqrt{5}$.

The surd $3\sqrt{20}$ can be written as $3\sqrt{4 \times 5}$. Simplifying further, it can be written as $3 \times 2\sqrt{5} = 6\sqrt{5}$

$$\therefore\ 3\sqrt{20} \div 2\sqrt{5} = \frac{3\sqrt{20}}{2\sqrt{5}} = \frac{6\sqrt{5}}{2\sqrt{5}}$$

$$= \underline{\underline{3}}$$

Next we will consider how expressions of the form $\frac{a}{\sqrt{b}}$ are simplified. Examples for such expressions are $\frac{3}{\sqrt{2}}$ and $\frac{4}{\sqrt{5}}$. Expression of this form has a square root term in the denominator.

Now let us consider how such an expression can be converted into an expression with an integer (or a rational number) in the denominator.

Example 6

Express $\frac{3}{\sqrt{2}}$ as a fraction with an integer in the denominator.

The method used here is, to multiply both the numerator and the denominator of $\frac{3}{\sqrt{2}}$ by $\sqrt{2}$

$$\frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \underline{\underline{\frac{3\sqrt{2}}{2}}}$$

This process is defined as **rationalizing the denominator**.

Example 7

Rationalise the denominator of $\frac{a}{\sqrt{b}}$

$$\frac{a}{\sqrt{b}} = \frac{a \times \sqrt{b}}{\sqrt{b} \times \sqrt{b}}$$

$$= \underline{\underline{\frac{a\sqrt{b}}{b}}}$$

Now let us consider how an expression involving surds is simplified.

Example 8

Simplify $4\sqrt{63} - 5\sqrt{7} - 8\sqrt{28}$.

$$4\sqrt{63} = 4 \times \sqrt{9 \times 7} = 4 \times 3\sqrt{7}$$
$$= 12\sqrt{7}$$
$$8\sqrt{28} = 8 \times \sqrt{4 \times 7} = 8 \times 2\sqrt{7}$$
$$= 16\sqrt{7}$$

Therefore $4\sqrt{63} - 5\sqrt{7} - 8\sqrt{28} = 12\sqrt{7} - 5\sqrt{7} - 16\sqrt{7}$

$$= -\underline{\underline{9\sqrt{7}}}$$

Let us consider how a more complex expression involving surds is simplified.

Example 9

Simplify $\frac{2\sqrt{6}}{\sqrt{2}} + \sqrt{75} - \frac{3}{\sqrt{12}}$

$$
\begin{aligned}
\frac{2\sqrt{6}}{\sqrt{2}} + \sqrt{75} - \frac{3}{\sqrt{12}} &= \frac{2\sqrt{2\times 3}}{\sqrt{2}} + \sqrt{25\times 3} - \frac{3}{\sqrt{4\times 3}} \\
&= \frac{2\sqrt{2}\times\sqrt{3}}{\sqrt{2}} + \sqrt{25\times 3} - \frac{3}{\sqrt{4\times 3}} \\
&= 2\sqrt{3} + 5\sqrt{3} - \frac{3}{2\sqrt{3}} \\
&= 7\sqrt{3} - \frac{3\times\sqrt{3}}{2\sqrt{3}\times\sqrt{3}} \\
&= 7\sqrt{3} - \frac{3\sqrt{3}}{2\times 3} \\
&= 7\sqrt{3} - \frac{\sqrt{3}}{2} \\
&= \frac{13\sqrt{3}}{2}
\end{aligned}
$$

Exercise 1.3

1. Convert the following entire surds into surds.

a. $\sqrt{20}$ **b.** $\sqrt{48}$ **c.** $\sqrt{72}$ **d.** $\sqrt{28}$

e. $\sqrt{80}$ **f.** $\sqrt{45}$ **g.** $\sqrt{75}$ **h.** $\sqrt{147}$

2. Convert the following surds into entire surds.

a. $2\sqrt{3}$ **b.** $2\sqrt{5}$ **c.** $4\sqrt{7}$ **d.** $5\sqrt{2}$ **e.** $6\sqrt{11}$

3. Simplify.

a. $\sqrt{2} + 5\sqrt{2} - 2\sqrt{2}$

b. $\sqrt{5} + 2\sqrt{7} + 2\sqrt{5} - 3\sqrt{7}$

c. $4\sqrt{3} + 5\sqrt{2} + 3\sqrt{5} - 3\sqrt{2} + 3\sqrt{5} - 2\sqrt{3}$

d. $6\sqrt{11} + 3\sqrt{7} - 2\sqrt{11} - 5\sqrt{7} + 4\sqrt{7}$

e. $8\sqrt{3} + 7\sqrt{7} - 2\sqrt{3} + 3\sqrt{7} - 3\sqrt{7}$

4. Rationalise the denominator of the following fractions.

a. $\dfrac{2}{\sqrt{5}}$ b. $\dfrac{5}{\sqrt{3}}$ c. $\dfrac{5}{\sqrt{7}}$ d. $\dfrac{12}{2\sqrt{3}}$ e. $\dfrac{27}{3\sqrt{2}}$

f. $\dfrac{3}{2\sqrt{5}}$ g. $\dfrac{3\sqrt{5}}{2\sqrt{7}}$ h. $\dfrac{2\sqrt{3}}{3\sqrt{2}}$ i. $\dfrac{3\sqrt{3}}{2\sqrt{5}}$

5. Simplify.

a. $3\sqrt{2} \times 2\sqrt{3}$ b. $5\sqrt{11} \times 3\sqrt{7}$ c. $\sqrt{5} \times 3\sqrt{3}$

d. $4\sqrt{7} \div 2\sqrt{14}$ e. $6\sqrt{27} \div 3\sqrt{3}$ f. $\sqrt{48} \div 5\sqrt{3}$

6. Simplify.

a. $2\sqrt{27} - 3\sqrt{3} + 4\sqrt{7} + 3\sqrt{28}$ b. $3\sqrt{63} - 2\sqrt{7} + 3\sqrt{27} + 3\sqrt{3}$

c. $2\sqrt{128} - 3\sqrt{50} + 2\sqrt{162} + \dfrac{4}{\sqrt{2}}$ d. $\sqrt{99} - 2\sqrt{44} + \dfrac{110}{\sqrt{11}}$

e. $\dfrac{\sqrt{20}}{2} - \sqrt{5}$

2 Indices and Logarithms I

By studying this lesson, you will be able to

- simplify expressions involving powers and roots and
- solve equations

using the laws of indices and logarithms.

Indices

Do the following exercise to revise what you have learned so far about indices and logarithms.

Review Exercise

1. Simplify and find the value.

a. $2^2 \times 2^3$ **b.** $(2^4)^2$ **c.** 3^{-2}

d. $\dfrac{5^3 \times 5^2}{5^5}$ **e.** $\dfrac{3^5 \times 3^2}{3^6}$ **f.** $(5^2)^2 \div 5^3$

g. $\dfrac{(2^2)^3 \times 2^4}{2^8}$ **h.** $\dfrac{5^{-3} \times 5^2}{5^0}$ **i.** $(5^2)^{-2} \times 5 \times 3^0$

2. Simplify.

a. $a^2 \times a^3 \times a$ **b.** $a^5 \times a \times a^0$ **c.** $(a^2)^3$

d. $(x^2)^3 \times x^2$ **e.** $(xy)^2 \times x^0$ **f.** $(2x^2)^3$

g. $\dfrac{2pq \times 3p}{6p^2}$ **h.** $2x^{-2} \times 5xy$ **i.** $\dfrac{(3a)^{-2} \times 4a^2b^2}{2ab}$

3. Simplify.

a. $\lg 25 + \lg 4$ **b.** $\log_2 8 - \log_2 4$

c. $\log_5 50 + \log_5 2 - \log_5 4$ **d.** $\log_a 5 + \log_a 4 - \log_a 2$

e. $\log_x 4 + \log_x 12 - \log_x 3$ **f.** $\log_p a + \log_p b - \log_p c$

4. Solve the following equations.

a. $\log_5 x = \log_5 4 + \log_5 2$ **b.** $\log_5 4 - \log_5 2 = \log_5 x$

c. $\log_a 2 + \log_a x = \log_a 10$ **d.** $\log_3 x + \log_3 10 = \log_3 5 + \log_3 6 - \log_3 2$

e. $\lg 5 - \lg x + \lg 8 = \lg 4$ **f.** $\log_x 12 - \log_5 4 = \log_5 3$

2.1 Fractional Indices of a Power

Square root of 4 can be written either using the radical symbol (square root symbol) as $\sqrt{4}$ or using powers as $4^{\frac{1}{2}}$.
Therefore, it is clear that $\sqrt{4} = 4^{\frac{1}{2}}$.

Let us consider another example, similar to the above. As $2 = 2^1$,

$$\begin{aligned} 2 \times 2 \times 2 &= 2^1 \times 2^1 \times 2^1 \\ &= 2^3 \\ &= 8 \end{aligned}$$

Third power of 2 is 8. Thus, the cube root of 8 is 2. This can be denoted symbolically as,

$$\sqrt[3]{8} = 2 \text{ or } 8^{\frac{1}{3}} = 2.$$

Therefore it is clear that $\sqrt[3]{8} = 8^{\frac{1}{3}}$.
Futhermore, if a is a positive real number, then

$$\sqrt{a} = a^{\frac{1}{2}},$$
$$\sqrt[3]{a} = a^{\frac{1}{3}} \text{ and}$$
$$\sqrt[4]{a} = a^{\frac{1}{4}}.$$

Thus, the general relationship between the radical symbol and the exponent (index) of a power can be expressed as follows.

$$\boxed{\sqrt[n]{a} = a^{\frac{1}{n}}}$$

The following examples demonstrate how the above relationship can be used to simplify expressions involving powers.

Example 1

1. Find the value.

(i) $\sqrt[3]{27}$ (ii) $(\sqrt{25})^{-2}$ (iii) $\sqrt[3]{3\frac{3}{8}}$

(i) $\sqrt[3]{27} = 27^{\frac{1}{3}}$

$= (3^3)^{\frac{1}{3}}$

$= 3^{3 \times \frac{1}{3}}$

$= \underline{\underline{3}}$

(ii) $(\sqrt{25})^{-2} = (25^{\frac{1}{2}})^{-2}$

$= \{(5^2)^{\frac{1}{2}}\}^{-2}$

$= (5^{2 \times \frac{1}{2}})^{-2}$

$= 5^{-2}$

$= \frac{1}{5^2}$

$= \underline{\underline{\frac{1}{25}}}$

(iii) $\sqrt[3]{3\frac{3}{8}} = \sqrt[3]{\frac{27}{8}}$

$= \left(\frac{27}{8}\right)^{\frac{1}{3}}$

$= \frac{(3^3)^{\frac{1}{3}}}{(2^3)^{\frac{1}{3}}}$

$= \frac{3^{3 \times \frac{1}{3}}}{2^{3 \times \frac{1}{3}}}$

$= \frac{3}{2}$

$= \underline{\underline{1\frac{1}{2}}}$

The following examples further investigate how the laws of indices are used to simplify algebraic expressions involving powers.

Example 2

Simplify and express the answer with positive exponents (indices).

(i) $\left(\sqrt{x}\right)^3$ (ii) $\left(\sqrt[3]{a}\right)^{-\frac{1}{2}}$ (iii) $\frac{1}{\sqrt{x^{-3}}}$

(i) $\left(\sqrt{x}\right)^3 = \left(x^{\frac{1}{2}}\right)^3$

$= x^{\frac{1}{2} \times 3}$

$= \underline{\underline{x^{\frac{3}{2}}}}$

(ii) $\left(\sqrt[3]{a}\right)^{-\frac{1}{2}} = \left(a^{\frac{1}{3}}\right)^{-\frac{1}{2}}$

$= a^{\frac{1}{3} \times -\frac{1}{2}}$

$= a^{-\frac{1}{6}}$

$= \underline{\underline{\frac{1}{a^{\frac{1}{6}}}}}$

(iii) $\frac{1}{\sqrt{x^{-3}}} = \frac{1}{(x^{-3})^{\frac{1}{2}}}$

$= \frac{1}{x^{-3 \times \frac{1}{2}}}$

$= \frac{1}{x^{-\frac{3}{2}}}$

$= \underline{\underline{x^{\frac{3}{2}}}}$

Example 3

Find the value. (i) $\left(\frac{27}{64}\right)^{\frac{2}{3}}$ (ii) $\left(\frac{16}{81}\right)^{-\frac{3}{4}}$

(i) $$\left(\frac{27}{64}\right)^{\frac{2}{3}} = \left(\frac{3^3}{4^3}\right)^{\frac{2}{3}}$$
$$= \left[\left(\frac{3}{4}\right)^3\right]^{\frac{2}{3}}$$
$$= \left(\frac{3}{4}\right)^{3\times\frac{2}{3}}$$
$$= \left(\frac{3}{4}\right)^2$$
$$= \underline{\underline{\frac{9}{16}}}$$

(ii) $$\left(\frac{16}{81}\right)^{-\frac{3}{4}} = \left(\frac{2^4}{3^4}\right)^{-\frac{3}{4}}$$
$$= \left(\frac{2}{3}\right)^{4\times-\frac{3}{4}}$$
$$= \left(\frac{2}{3}\right)^{-3}$$
$$= \left(\frac{3}{2}\right)^{3}$$
$$= \frac{27}{8}$$
$$= \underline{\underline{3\frac{3}{8}}}$$

Let us now consider a slightly complex example: $\left(\frac{125}{64}\right)^{-\frac{1}{3}} \times \left(\sqrt[5]{32}\right)^3 \times 3^0$

$$\left(\frac{125}{64}\right)^{-\frac{1}{3}} \times \left(\sqrt[5]{32}\right)^3 \times 3^0 = \left(\frac{5^3}{2^6}\right)^{-\frac{1}{3}} \times \left(32^{\frac{1}{5}}\right)^3 \times 1$$
$$= \left(\frac{2^6}{5^3}\right)^{\frac{1}{3}} \times \left(2^{5\times\frac{1}{5}}\right)^3$$
$$= \frac{2^{6\times\frac{1}{3}}}{5^{3\times\frac{1}{3}}} \times 2^3$$
$$= \frac{2^2}{5} \times 2^3$$
$$= \frac{2^5}{5}$$
$$= \frac{32}{5}$$
$$= \underline{\underline{6\frac{2}{5}}}$$

Example 4

Simplify: $\dfrac{\sqrt[3]{343x^{3/2}}}{x}$

$$\begin{aligned}\frac{\sqrt[3]{343x^{3/2}}}{x} &= \left(343x^{\frac{3}{2}}\right)^{\frac{1}{3}} \div x \\ &= 343^{\frac{1}{3}} \times \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} \div x \\ &= (7^3)^{\frac{1}{3}} \times \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} \div x \\ &= 7^1 \times x^{\frac{1}{2}} \div x \\ &= 7 \times x^{\frac{1}{2}-1} \\ &= 7 \times x^{-\frac{1}{2}} \\ &= \underline{\underline{\frac{7}{x^{\frac{1}{2}}}}}\end{aligned}$$

Exercise 2.1

1. Express the following using the radical symbol.

a. $p^{\frac{1}{3}}$ **b.** $a^{\frac{2}{3}}$ **c.** $x^{-\frac{2}{3}}$

d. $m^{\frac{4}{5}}$ **e.** $y^{-\frac{3}{4}}$ **f.** $x^{-\frac{5}{3}}$

2. Write using positive exponents (indices).

a. $\sqrt{m^{-1}}$ **b.** $\sqrt[3]{x^{-1}}$ **c.** $\sqrt[5]{p^{-2}}$ **d.** $(\sqrt{a})^{-3}$ **e.** $\sqrt[4]{x^{-3}}$

f. $(\sqrt[3]{p})^{-5}$ **g.** $\dfrac{1}{\sqrt{x^{-3}}}$ **h.** $\dfrac{1}{\sqrt[3]{a^{-2}}}$ **i.** $2\sqrt[3]{x^{-2}}$ **j.** $\dfrac{1}{3\sqrt{a^{-5}}}$

3. Find the value.

a. $\sqrt{25}$ **b.** $\sqrt[4]{16}$ **c.** $(\sqrt{4})^5$

d. $(\sqrt[3]{27})^2$ **e.** $\sqrt[4]{81^3}$ **f.** $\sqrt[3]{1000^2}$

g. $\left(\frac{27}{125}\right)^{\frac{2}{3}}$ **h.** $\left(\frac{81}{10000}\right)^{\frac{3}{4}}$ **i.** $\left(\frac{1}{64}\right)^{-\frac{5}{6}}$

j. $\left(\frac{27}{64}\right)^{-\frac{2}{3}}$ **k.** $(0.81)^{\frac{3}{2}}$ **l.** $(0.125)^{-\frac{2}{3}}$

m. $\left(\frac{4}{25}\right)^{\frac{1}{2}} \times \left(\frac{3}{4}\right)^{-1} \times 2^0$ **n.** $\left(\frac{9}{100}\right)^{-\frac{3}{2}} \times \left(\frac{4}{25}\right)^{\frac{3}{2}}$ **o.** $(27)^{1\frac{1}{3}} \times (81)^{-1\frac{1}{4}}$

p. $\left(11\frac{1}{9}\right)^{-\frac{1}{2}} \times \left(6\frac{1}{4}\right)^{-\frac{3}{2}}$ **q.** $(0.125)^{-\frac{1}{3}} \times (0.25)^{\frac{3}{2}}$ **r.** $(\sqrt[3]{8})^2 \times \sqrt[4]{16^3}$

4. Simplify and express using positive indices.

a. $\sqrt[3]{a^{-1}} \div \sqrt[3]{a}$ **b.** $\sqrt[5]{a^{-3}} \div \sqrt[5]{a^7}$ **c.** $\sqrt[3]{a^2} \div \sqrt[3]{a^3}$

d. $\left(\sqrt[3]{x^5}\right)^{\frac{1}{2}} \times \sqrt[6]{x^{-5}}$ **e.** $\left\{\left(\sqrt{a^3}\right)^{-2}\right\}^{\frac{-1}{2}}$ **f.** $\left(\sqrt{x^2y^2}\right)^{-6}$

g. $\sqrt{\frac{4a^{-2}}{9x^2}}$ **h.** $\left(\sqrt[3]{27x^3}\right)^{-2}$ **i.** $\left(\frac{xy^{-1}}{\sqrt{x^5}}\right)^{-2}$

2.2 Solving Equations with Indices

$2^x = 2^3$ is an equation. Because the bases of the powers on either side of the equal sign are equal, the exponents must be equal. Thus, from $2^x = 2^3$, we can conclude that $x = 3$.
Similarly, on either side of the equation $x^5 = 2^5$ are powers with equal exponents.

Because the indices are equal, the bases are also equal. Therefore, from $x^5 = 2^5$ we can conclude that $x = 2$. If $x^2 = 3^2$ then the indices are equal but in this case, both $+ 3$ and $- 3$ are solutions. The reason for two solutions arising is because the exponent " 2 " is an even number. In this lesson, we will only consider powers with

a positive base. Thus, in expressions of the form x^m, $x > 0$.
There is a special property of powers of 1. All powers of 1 are equal to 1. That is, for any m, $1^m = 1$.

Let us summarise the above observations.

For $x > 0$, $y > 0$, $y \neq 1$ and $x \neq 1$,

if $x \neq 0$ $x^m = x^n$, then $m = n$.
if $m \neq 0$ and $x^m = y^m$, then $x = y$.

Let us use these rules to solve equations with indices.

Example 1

Solve.

(i) $4^x = 64$ (ii) $x^3 = 343$ (iii) $3 \times 9^{2x-1} = 27^{-x}$

(i) $4^x = 64$

$4^x = 4^3$

$\therefore \underline{\underline{x = 3}}$

(ii) $x^3 = 343$

$x^3 = 7^3$

$\therefore \underline{\underline{x = 7}}$

(iii)
$$\begin{aligned}
3 \times 9^{2x-1} &= 27^{-x} \\
3 \times (3^2)^{2x-1} &= (3)^{3(-x)} \\
3 \times 3^{2(2x-1)} &= 3^{-3x} \\
3^{1+4x-2} &= 3^{-3x} \\
\therefore\ 1 + 4x - 2 &= -3x \\
4x + 3x &= 2 - 1 \\
7x &= 1 \\
x &= \underline{\underline{\frac{1}{7}}}
\end{aligned}$$

Exercise 2.2

1. Solve each of the following equations.

a. $3^x = 9$ **b.** $3^{x+2} = 243$

c. $4^{3x} = 32$ **d.** $2^{5x-2} = 8^x$

e. $8^{x-1} = 4^x$ **f.** $x^3 = 216$

g. $2\sqrt{x} = 6$ **h.** $\sqrt[3]{2x^2} = 2$

2. Solve each of the following equations.

a. $2^x \times 8^x = 256$ **b.** $8 \times 2^{x-1} = 4^{x-2}$

c. $5 \times 25^{2x-1} = 125$ **d.** $3^{2x} \times 9^{3x-2} = 27^{-3x}$

e. $4^x = \frac{1}{64}$ **f.** $(3^x)^{-\frac{1}{2}} = \frac{1}{27}$

g. $3^{4x} \times \frac{1}{9} = 9^x$ **h.** $x^2 = \left(\frac{1}{8}\right)^{-\frac{2}{3}}$

2.3 Laws of logarithms

We know that, using the laws of logarithms, we can write

$\log_2 (16 \times 32) = \log_2 16 + \log_2 32$ and $\log_2 (32 \div 16) = \log_2 32 - \log_2 16$. These laws, in general, can be written as follows.

$$\log_a (mn) = \log_a m + \log_a n$$

$$\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Let us learn another law of a similar type.

Consider $\log_5 125^4$ as an example.

$$\begin{aligned} \log_5 125^4 &= \log_5 (125 \times 125 \times 125 \times 125) \\ &= \log_5 125 + \log_5 125 + \log_5 125 + \log_5 125 \\ &= 4 \log_5 125 \end{aligned}$$

Similarly,

$$\log_{10} 10^5 = 5 \log_{10} 10 \text{ and}$$

$$\log_3 5^2 = 2 \log_3 5$$

This observation can be written, in general, as the following logarithmic law.

$$\boxed{\log_a m^r = r \log_a m}$$

This law is even valid for expressions with fractional indices. Given below are a few examples, where this law is applied to powers with fractional indices.

$$\log_2 3^{\frac{1}{2}} = \frac{1}{2} \log_2 3$$

$$\log_5 7^{\frac{2}{3}} = \frac{2}{3} \log_5 7$$

The following examples consider how all the laws of logarithms that you have learned so far, including the above, are used.

Example 1

Evaluate.

(i) $\lg 1000$ (ii) $\log_4 \sqrt[3]{64}$ (iii) $2 \log_2 2 + 3 \log_2 4 - 2 \log_2 8$

(i) $$\begin{aligned} \lg 1000 &= \lg 10^3 \\ &= 3 \lg 10 \\ &= 3 \times 1 \quad \text{(because } \lg 10 = 1) \\ &= \underline{\underline{3}} \end{aligned}$$

(ii) $\log_4 \sqrt[3]{64} = \log_4 64^{\frac{1}{3}}$

$= \frac{1}{3} \log_4 64$

$= \frac{1}{3} \log_4 4^3$

$= \frac{1}{3} \times 3 \log_4 4$

$= \log_4 4$

$= \underline{\underline{1}}$

(iii) $2 \log_2 2 + 3 \log_2 4 - 2\log_2 8 = 2\log_2 2 + 3 \log_2 2^2 - 2\log_2 2^3$

$= \log_2 2^2 + \log_2 (2^2)^3 - \log_2 (2^3)^2$

$= \log_2 \left(\frac{2^2 \times (2^2)^3}{(2^3)^2} \right)$

$= \log_2 \left(\frac{2^2 \times 2^6}{2^6} \right)$

$= \log_2 2^2$

$= 2 \log_2 2$

$= \underline{\underline{2}}$

Example 2

Solve.

(i) $2\lg 8 + 2\lg 5 = \lg 4^3 + \lg x$

$\therefore \lg x = 2\lg 8 + 2\lg 5 - \lg 4^3$

$= \lg 8^2 + \lg 5^2 - \lg 4^3$

$\therefore \lg x = \lg \left(\frac{8^2 \times 5^2}{4^3} \right)$

$\therefore \lg x = \lg 25$

$\therefore \underline{\underline{x = 25}}$

(ii) $2\log_b 3 + 3\log_b 2 - \log_b 72 = \frac{1}{2}\log_b x$

$\therefore\ 2\log_b 3 + 3\log_b 2 - \log_b 72 = \frac{1}{2}\log_b x$

$\therefore\ \log_b 3^2 + \log_b 2^3 - \log_b 72 = \log_b x^{\frac{1}{2}}$

$\therefore\ \log_b\left(\frac{3^2\times 2^3}{72}\right) = \log_b x^{\frac{1}{2}}$

$\therefore\ \frac{3^2\times 2^3}{72} = x^{\frac{1}{2}}$

$\therefore\ 1^2 = (x^{\frac{1}{2}})^2$

$\therefore\ 1 = x^1$

$\therefore\ \underline{\underline{x = 1}}$

Example 3

Verify: $\log_5 75 - \log_5 3 = \log_5 40 - \log_5 8 + 1$

$$\text{Left Side} = \log_5 \frac{75}{3}$$

$$= \log_5 25$$

$$= \log_5 5^2$$

$$= 2$$

$$\text{Right Side} = \log_5 40 - \log_5 8 + 1$$

$$= \log_5 \frac{40}{8} + 1$$

$$= \log_5 5 + 1$$

$$= 1 + 1$$

$$= 2$$

$$\therefore\ \log_5 75 - \log_5 3 = \log_5 40 - \log_5 8 + 1$$

Use the laws of logarithms to do the following exercise.

Exercise 2.3

1. Evaluate.

a. $\log_2 32$ **b.** $\lg 10000$ **c.** $\frac{1}{3}\log_3 27$

d. $\frac{1}{2}\log_5 \sqrt{25}$ **e.** $\log_3 \sqrt[4]{81}$ **f.** $3\log_2 \sqrt[3]{8}$

2. Simplify each of the following expressions and find the value.

a. $2\log_2 16 - \log_2 8$

b. $\lg 80 - 3\lg 2$

c. $2\lg 5 + 3\lg 2 - \lg 2$

d. $\lg 75 - \lg 3 + \lg 28 - \lg 7$

e. $\lg 18 - 3\lg 3 + \frac{1}{2}\lg 9 + \lg 5$

f. $4\lg 2 + \lg \frac{15}{4} - \lg 6$

g. $\lg \frac{1}{256} - \lg \frac{125}{4} - 3\lg \frac{1}{20}$

h. $\log_3 27 + 2\log_3 3 - \log_3 3$

i. $\lg \frac{12}{5} + \lg \frac{25}{21} - \lg \frac{2}{7}$

j. $\lg \frac{3}{4} - 2\lg \frac{3}{10} + \lg 12 - 2$

3. Solve the following equations.

a. $\log x + \lg 4 = \lg 8 + \lg 2$

b. $4\lg 2 + 2\lg x + \lg 5 = \lg 15 + \lg 12$

c. $3\lg x + \log 96 = 2\lg 9 + \lg 4$

d. $\lg x = \frac{1}{2}(\lg 25 + \lg 8 - \lg 2)$

e. $3\lg x + 2\lg 8 = \lg 48 + \frac{1}{2}\lg 25 - \lg 30$

f. $\lg 125 + 2\lg 3 = 2\lg x + \lg 5$

Summary

- $\sqrt[n]{a} = a^{\frac{1}{n}}$

- If $x > 0, y > 0$ and $x \neq 1, y \neq 1$

  $x \neq 0$ and $x^m = x^n$, then $m = n$.
  $m \neq 0$ and $x^m = y^m$, then $x = y$.

- $\log_a m^r = r\log_a m$

Miscellaneous Exercise

1. Find the value.

a. $(\sqrt[3]{8})^2 \times \dfrac{1}{\sqrt[3]{27}}$

b. $(\sqrt{125})^3 \times \dfrac{1}{\sqrt{20}} \times 10$

c. $\dfrac{32^{-\frac{2}{5}} \times 216^{\frac{2}{3}}}{81^{\frac{3}{4}} \times \sqrt[3]{8^0} \times \sqrt[3]{27^{-2}}}$

d. $\sqrt{\dfrac{18 \times 5^2}{8}}$

e. $\left(\dfrac{1}{8}\right)^{-\frac{1}{3}} \times 5^{-2} \times 100$

f. $27^{\frac{2}{3}} - 16^{\frac{3}{4}}$

2. Simplify and express using positive indices.

a. $\sqrt{a^2 b^{-\frac{1}{2}}}$

b. $(x^{-4})^{\frac{1}{2}} \times \dfrac{1}{\sqrt{x^{-3}}}$

c. $(x^{\frac{1}{2}} - x^{-\frac{1}{2}})\ (x^{\frac{1}{2}} + x^{-\frac{1}{2}})$

d. $(x \div \sqrt[n]{x})^n$

e. $\left[\left(\sqrt{a^3}\right)^{-2}\right]^{\frac{1}{2}}$

3. Verify the following.

a. $\lg\left(\dfrac{217}{38} \div \dfrac{31}{266}\right) = 2 \lg 7$

b. $\dfrac{1}{2} \lg 9 + \lg 2 = 2 \lg 3 - \lg 1.5$

c. $\log_3 24 + \log_3 5 - \log_3 40 = 1$

d. $\lg 26 + \lg 119 - \lg 51 - \lg 91 = \lg 2 - \lg 3$

e. $2 \log_a 3 + \log_a 20 - \log_a 36 = \log_a 10 - \log_a 2$

3 Indices and Logarithms II

By studying this lesson you will be able to

- use the table of logarithms to simplify expressions involving products and quotients of powers and roots of numbers between 0 and 1.
- identify the two keys $\boxed{\wedge}$ and $\boxed{\sqrt{\ }}$ on a scientific calculater, and simplify expressions involving decimal numbers, powers and roots using a scientific calculator.

Logarithms

$10^3 = 1000$ can be written using logarithms as $\log_{10} 1000 = 3$. As a convention we write "lg" instead of "$\log_{10}$ ". Now we can express the above expression as $\lg 1000 = 3$. It is important to mention the base if it is other than 10.

For example,

$\log_5 25 = 2$ because $5^2 = 25$,
$\lg 1 = 0$ because $10^0 = 1$, and
$\lg 10 = 1$ because $10^1 = 10$.

The logarithm of any positive number can be found using the table of logarithms. Do the following exercise to refresh the memory on using logarithms to simplify expressions involving multiplications and divisions of numbers.

Review Exercise

1. Complete the following tables.

(i)

Number	Scientific notation	Logarithm		Logarithm
		Characteristic	Mantissa	
73.45	7.345×10^1	1	0.8660	1.8660
8.7				
12.5				
725.3				
975				

(ii)

Logarithm	Logarithm		Scientific notation	Number
	Characteristic	Mantissa		
1.5492				
2.9059				
1.4036				
2.8798				
3.4909				

2. Use the table of logarithms to fill in the blanks.

a. lg 5.745 = 0.7593, therefore 5.745 = $10^{0.7593}$
b. lg 9.005 = ..., therefore 9.005 = $10^{\cdots\cdots}$
c. lg 82.8 = ..., therefore 82.8 = $10^{\cdots\cdots}$
d. lg 74.01 = ..., therefore 74.01 = $10^{\cdots\cdots}$
e. lg 853.1 = ..., therefore 853.1 = $10^{\cdots\cdots}$
f. antilog 0.7453 = 5.562, therefore 5.562 = $10^{0.7453}$
g. antilog 0.0014 = ..., therefore .... = $10^{0.0014}$
h. antilog 1.9251 = ..., therefore .... = $10^{1.9251}$
i. antilog 2.4374 = ..., therefore .... = $10^{2.4374}$
j. antilog 3.2001 = ..., therefore .... = $10^{3.2001}$

3. Fill in the blanks and find the value of P.

(i) In terms of logarithms

$$P = \frac{27.32 \times 9.8}{11.5}$$

$$\begin{aligned}
\lg P &= \lg \ldots\ldots + \lg \ldots\ldots - \lg \ldots\ldots \\
&= \ldots\ldots + \ldots\ldots - \ldots\ldots \\
&= \ldots\ldots \\
\therefore\ P &= \text{antilog } \ldots\ldots \\
&= \underline{\underline{\ldots\ldots\ldots}}
\end{aligned}$$

(ii) Using indices

$$\begin{aligned}
P &= \frac{27.32 \times 9.8}{11.5} \\
&= \frac{10^{\cdots} \times 10^{\cdots}}{10^{\cdots}} \\
&= \frac{10^{\cdots}}{10^{\cdots}} \\
&= 10^{\cdots} \\
&= \ldots\ldots \times 10^{\cdots} \\
&= \underline{\underline{\ldots\ldots\ldots}}
\end{aligned}$$

4. Simplify the expressions using logarithms.

a. 14.3×95.2 **b.** $2.575 \times 9.27 \times 12.54$ **c.** $\dfrac{9.87 \times 7.85}{4.321}$

3.1 Logarithms of decimal numbers less than one

Let us now consider how to use the table of logarithms to obtain the logarithms of numbers between 0 and 1, by paying close attention to how we obtained the logarithms of numbers greater than 1. For this purpose, carefully investigate the following table.

Number	Scientific Notation	Logarithm		Logarithm
		Characteristic	Mantissa	
5432	5.432×10^3	3	0.7350	3.7350
543.2	5.432×10^2	2	0.7350	2.7350
54.32	5.432×10^1	1	0.7350	1.7350
5.432	5.432×10^0	0	0.7350	0.7350
0.5432	5.432×10^{-1}	-1	0.7350	$\overline{1}.7350$
0.05432	5.432×10^{-2}	-2	0.7350	$\overline{2}.7350$
0.005432	5.432×10^{-3}	-3	0.7350	$\overline{3}.7350$
0.0005432	5.432×10^{-4}	-4	0.7350	$\overline{4}.7350$

According to the above table, the characteristic of the logarithm of numbers inbetween 0 and 1, coming after 5.432 in the first column, are negative. Even though the characteristic is negative, the mantissa of the logarithm, which is found using the table, is a positive number. The symbol "$-$" is used above the whole part to indicate that only the characteristic is negative. It is read as "**bar**".

For example, $\overline{2}.3725$ is read as "bar two point three, seven, two, five". Moreover, what is represented by $\overline{2}.3725$ is $-2 + 0.3725$.

The characteristic of the logarithm of a number between 0 and 1 is negative. The characteristic of the logarithm of such a number can be obtained either by writing it in scientific notation or by counting the number of zeros after the decimal point.

The characteristic of the logarithm can be obtained by adding one to the number of zeros after the decimal point (and before the next non-zero digit) and taking its negative value. Observe it in the above table too.

Example

0.004302 Number of zeros after the decimal point and before the next non-zero digit is 2. Therefore characteristic of the logarithm is $\overline{3}$

0.04302	Number of zeros after the decimal point is 1; thus the characteristic of the logarithm is $\overline{2}$
0.4302	Number of zeros after the decimal point is 0; thus the characteristic of the logarithm is $\overline{1}$

Therefore, $\lg 0.004302 = \overline{3}.6337$.

When written using indices, it is;
$0.004302 = 10^{\overline{3}.6337}$. This can also be written as, $0.004302 = 10^{-3} \times 10^{0.6337}$.

Do the following exercise to practice taking logarithms of numbers between 0 and 1.

Exercise 3.1

1. For each of the following numbers, write the characteristic of its logarithm.

a. 0.9843 **b.** 0.05 **c.** 0.0725
d. 0.0019 **e.** 0.003141 **f.** 0.000783

2. Find the value.

a. lg 0.831 **b.** lg 0.01175 **c.** lg 0.0034
d. lg 0.009 **e.** lg 0.00005 **f.** lg 0.00098

3. Express each of the following numbers as a power of 10.

a. 0.831 **b.** 0.01175 **c.** 0.0034
d. 0.009 **e.** 0.00005 **f.** 0.00098

3.2 Number corresponding to a logarithm (antilog)

Let us recall how the antilog of a number greater than 1 is obtained.

$$\begin{aligned}\text{antilog } 2.7421 &= 5.522 \times 10^2 \\ &= 552.2\end{aligned}$$

When a number is written in scientific form, the index of the power of 10 is the characteristic of the logarithm of that number. The characteristic of the logarithm indicates the number of places that the decimal point needs to be shifted when taking the antilog.

Thus, we obtained 552.2 by shifting the decimal point of 5.522 two places to the right. However, when the characteristic is negative the decimal point is shifted to the left side.

antilog $\overline{2}.7421 = 5.522 \times 10^{-2}$ (Decimal point needs to be shifted 2 places to the left)

$= 0.05522$ (Because of bar 2, there is one 0 after the decimal point)

antilog $\overline{1}.7421 = 5.522 \times 10^{-1}$ (Decimal place needs to be shifted one place to the left)

$= 0.5522$ (Because of bar 1, there are no zeros after the decimal place)

Exercise 3.2

1. Express each of the following numbers, given in the scientific form, in decimal form.

a. 3.37×10^{-1} **b.** 5.99×10^{-3} **c.** 6.0×10^{-2}
d. 5.745×10^{0} **e.** 9.993×10^{-4} **f.** 8.777×10^{-3}

2. Find the value using the logarithmic table.

a. antilog $\overline{2}.5432$ **b.** antilog $\overline{1}.9321$ **c.** antilog 0.9972
d. antilog $\overline{4}.5330$ **e.** antilog $\overline{2}.0000$ **f.** antilog $\overline{3}.5555$

3.3 Addition and subtraction of logarithms with negative characteristics

(a) Addition

The mantissa of a logarithm is obtained from the table of logarithms and is always positive. But we now know that, the characteristic can be positive, negative or zero. In $\overline{2}.5143$, the mantissa, .5143, is positive and the characteristic, $\overline{2}$, is negative. When adding or subtracting such numbers, it is important to simplify the characteristic and the mantissa separately.

Example 1

Simplify and express the answer in log form.

(i) $\overline{2}.5143 + \overline{1}.2375$ (ii) $\overline{3}.9211 + 2.3142$ (iii) $\overline{3}.8753 + \overline{1}.3475$

(i) $\overline{2}.5143 + \overline{1}.2375$

$$
\begin{aligned}
&= -2 + 0.5143 + (-1) + 0.2375 \\
&= (-2 - 1) + (0.5143 + 0.2375) \\
&= -3 + 0.7518 \\
&= \underline{\underline{\overline{3}.7518}}
\end{aligned}
$$

(ii) $\bar{3}.9211 + 2.3142$ $= -3 + 0.9211 + 2 + 0.3142$
$= (-3 + 2) + (0.9211 + 0.3142)$
$= -1 + 1.2353$
$= -1 + 1 + 0.2353$
$= \underline{\underline{0.2353}}$

(iii) $\bar{3}.8753 + 1.3475$ $= -3 + 0.8753 + 1 + 0.3475$
$= (-3 + 1) + (0.8753 + 0.3475)$
$= -2 + 1.2228$
$= -2 + 1 + 0.2228$
$= \underline{\underline{\bar{1}.2228}}$

(b) Subtraction

As in addition, logarithms should be subtracted from right to left, remembering that the mantissa is positive.

Example 2

Simplify and express the answer in log form.

(i) $\bar{2}.5143 - 1.3143 = -2 + 0.5143 - (1 + 0.3143)$
$= -2 + 0.5143 - 1 - 0.3143$
$= -2 - 1 + 0.5143 - 0.3143$
$= -3 + 0.2000$
$= \underline{\underline{\bar{3}.2000}}$

(ii) $2.5143 - \bar{1}.9143 = 2 + 0.5143 - (-1 + 0.9143)$
$= 2 + 0.5143 + 1 - 0.9143$
$= 3 - 0.4000$
$= \underline{\underline{2.6000}}$

(iii) $0.2143 - \bar{1}.8143 = 0.2143 - (-1 + 0.8143)$
$= 0.2143 + 1 - 0.8143$
$= 1 - 0.6000$
$= \underline{\underline{0.4}}$

(iv) $\bar{2}.5143 - \bar{1}.9143 = -2 + 0.5143 - (-1 + 0.9143)$
$= -2 + 0.5143 + 1 - 0.9143$
$= -2 + 1 + 0.5143 - 0.9143$
$= -1 - 0.4000$

In the above example, the decimal part is negative. Because we need the decimal part of a logarithm to be positive, we will use a trick as follows to make it positive.

$$-1-0.4=-1-1+1-0.4 \quad \text{(Because } -1+1=0\text{, we have not changed the value.)}$$
$$=-2+0.6$$
$$=\overline{2}.6$$

Actually what we have done is add -1 to the characteristic and $+1$ to the mantissa.

Note: We could have avoided getting a negative decimal part by doing the simplification as follows.

$$-2+0.5143+1-0.9143=-2+1.5143-0.9143=-2+0.6=\overline{2}.6$$

Exercise 3.3

1. Simplify

a. $0.7512+\overline{1}.3142$ **b.** $\overline{1}.3072+\overline{2}.2111$ **c.** $\overline{2}.5432+\overline{1}.9513$

d $\overline{3}.9121+\overline{1}.5431$ **e.** $0.7532+\overline{3}.8542$ **f.** $\overline{1}.8311+\overline{2}.5431+1.3954$

g. $3.8760-\overline{2}.5431$ **h.** $\overline{2}.5132-\overline{1}.9332$ **i.** $\overline{3}.5114-\overline{2}.4312$

j $\overline{2}.9372-1.5449$ **k.** $0.7512+\overline{1}.9431$ **l.** $\overline{1}.9112-\overline{3}.9543$

2. Simplify and express in log form.

a. $\overline{1}.2513+0.9172-\overline{1}.514$ **b.** $\overline{3}.2112+2.5994-\overline{1}.5004$

c. $\overline{3}.2754+\overline{2}.8211-\overline{1}.4372$ **d.** $0.8514-\overline{1}.9111-\overline{2}.3112$

e. $\overline{3}.7512-(0.2511+\overline{1}.8112)$ **f.** $\overline{1}.2572+3.9140-\overline{1}.1111$

3.4 Simplification of numerical expressions using the table of logarithms

The following examples show how numerical computations are done using the given logarithmic rules.

1. $\log_a (P\times Q)=\log_a P+\log_a Q$

2. $\log_a \left(\frac{P}{Q}\right)=\log_a P-\log_a Q$

Example 1

Simplify using the table of logarithms and logarithmic rules.

a. 43.85×0.7532 **b.** 0.0034×0.8752

c. $0.0875 \div 18.75$ **d.** $0.3752 \div 0.9321$

Two methods of simplifying are shown below.

Method 1:

a. 43.85×0.7532

Take $P = 43.85 \times 0.7532$

$$\begin{aligned} \text{Then } \lg P &= \lg (43.85 \times 0.7532) \\ &= \lg 43.85 + \lg 0.7532 \\ &= 1.6420 + \bar{1}.8769 \\ &= 1 + 0.6420 - 1 + 0.8769 \\ &= 1.5189 \\ \therefore \quad P &= \text{antilog } 1.5189 \\ &= \underline{\underline{33.03}} \end{aligned}$$

Method 2:

Simplifying using indices

$$\begin{aligned} & 43.85 \times 0.7532 \\ &= 10^{1.6420} \times 10^{\bar{1}.8769} \\ &= 10^{1.5189} \\ &= 3.303 \times 10^{1} \\ &= \underline{\underline{33.03}} \end{aligned}$$

b. 0.0034×0.8752

Take $P = 0.0034 \times 0.8752$.

Then,

$$\begin{aligned} \lg P &= \lg (0.0034 \times 0.8752) \\ &= \lg 0.0034 + \lg 0.8752 \\ &= \bar{3}.5315 + \bar{1}.9421 \\ &= -3 + 0.5315 - 1 + 0.9421 \\ &= -4 + 1.4736 \\ &= -4 + 1 + 0.4736 \\ &= -3 + 0.4736 \\ &= \bar{3}.4736 \\ \therefore \quad P &= \text{antilog } \bar{3}.4736 \\ &= \underline{\underline{0.002975}} \end{aligned}$$

Simplifying using indices

$$\begin{aligned} & 0.0034 \times 0.8752 \\ &= 10^{\bar{3}.5315} \times 10^{\bar{1}.9421} \\ &= 10^{\bar{3}.4736} \\ &= 2.975 \times 10^{-3} \\ &= \underline{\underline{0.002975}} \end{aligned}$$

c. $0.0875 \div 18.75$

Take $P = 0.0875 \div 18.75$

Then, $\lg P = \lg (0.0875 \div 18.75)$

$= \lg 0.0875 - \lg 18.75$

$= \overline{2}.9420 - 1.2730$

$= -2 + 0.9420 - 1 - 0.2730$

$= -3 + 0.6690$

$= \overline{3}.6690$

$\therefore \quad P = \text{antilog } \overline{3}.6690$

$= \underline{\underline{0.004666}}$

Simplifying using indices

$0.0875 \div 18.75$

$= 10^{\overline{2}.9420} \div 10^{1.2730}$

$= 10^{\overline{2}.9420 - 1.2730}$

$= 10^{\overline{3}.6690}$

$= 4.666 \times 10^{-3}$

$= \underline{\underline{0.004666}}$

d. $0.3752 \div 0.9321$

Take $P = 0.3752 \div 0.9321$

Then, $\lg P = \lg (0.3752 \div 0.9321)$

$= \lg 0.3752 - \lg 0.9321$

$= \overline{1}.5742 - \overline{1}.9694$

$= -1 + 0.5742 - (-1 + 0.9694)$

$= -1 + 0.5742 + 1 - 0.9694$

$= -1 + 0.5742 + 0.0306$

$= -1 + 0.6048$

$= \overline{1}.6048$

$\therefore \quad P = \text{antilog } \overline{1}.6048$

$= \underline{\underline{0.4026}}$

Simplifying using indices

$0.3752 \div 0.9321$

$= 10^{\overline{1}.5742} \div 10^{\overline{1}.9694}$

$= 10^{\overline{1}.5742 - \overline{1}.9694}$

$= 10^{\overline{1}.6048}$

$= 4.026 \times 10^{-1}$

$= \underline{\underline{0.4026}}$

Example 2

Simplify using the table of logarithms.

$$\frac{8.753 \times 0.02203}{0.9321}$$

Take $P = \dfrac{8.753 \times 0.02203}{0.9321}$.

$$\begin{aligned}
\text{Then, } \lg P &= \lg\left(\frac{8.753 \times 0.02203}{0.9321}\right)\\
&= \lg 8.753 + \lg 0.02203 - \lg 0.9321\\
&= 0.9421 + \bar{2}.3430 - \bar{1}.9694\\
&= 0.9421 - 2 + 0.3430 - \bar{1}.9694\\
&= \bar{1}.2851 - \bar{1}.9694\\
&= -1 + 0.2851 - (-1 + 0.9694)\\
&= -1 + 0.2851 + 1 - 0.9694\\
&= \bar{1}.3157\\
\therefore\ P &= \text{antilog } \bar{1}.3157\\
&= \underline{\underline{0.2068}}
\end{aligned}$$

Simplifying using indices

$$\begin{aligned}
&\frac{8.753 \times 0.02203}{0.9321}\\
&= \frac{10^{0.9421} \times 10^{\bar{2}.3430}}{10^{\bar{1}.9694}}\\
&= \frac{10^{\bar{1}.2851}}{10^{\bar{1}.9694}}\\
&= 10^{\bar{1}.2851 - \bar{1}.9694}\\
&= 10^{\bar{1}.3157}\\
&= 2.067 \times 10^{-1}\\
&= \underline{\underline{0.2068}}
\end{aligned}$$

Exercise 3.4

Find the value using the table of logarithms.

1. **a.** 5.945×0.782 **b.** 0.7453×0.05921 **c.** 0.0085×0.0943

d. $5.21 \times 0.752 \times 0.093$ **e.** $857 \times 0.008321 \times 0.457$ **f.** $0.123 \times 0.9857 \times 0.79$

2.

a. $7.543 \div 0.9524$ **b.** $0.0752 \div 0.8143$ **c.** $0.005273 \div 0.0078$

d. $0.9347 \div 8.75$ **e.** $0.0631 \div 0.003921$ **f.** $0.0752 \div 0.0008531$

3.

a. $\dfrac{8.247 \times 0.1973}{0.9875}$ **b.** $\dfrac{9.752 \times 0.0054}{0.09534}$ **c.** $\dfrac{79.25 \times 0.0043}{0.3725}$

d. $\dfrac{0.7135 \times 0.4391}{0.0059}$ **e.** $\dfrac{5.378 \times 0.9376}{0.0731 \times 0.471}$ **f.** $\dfrac{71.8 \times 0.7823}{23.19 \times 0.0932}$

3.5 Multiplication and division of a logarithm of a number by a whole number

We know that the characteristic of a number greater than one is positive. Multiplying or dividing such a logarithm by a number can be done in the usual way.

We know that the characteristic of the logarithm of a number between 0 and 1 is negative. $\bar{3}.8247$ is such a logarithm. When multiplying or dividing a logarithm with a negative characteristic by a number, we simplify the characteristic and the mantissa separately.

Multiplication of a logarithm by a whole number

Example 1

Simplify.

a. 2.8111×2 **b.** $\bar{2}.7512 \times 3$ **c.** $\bar{1}.9217 \times 3$

a.

$$\begin{aligned} & 2.8111 \times 2 \\ &= \underline{\underline{5.6222}} \end{aligned}$$

b.

$$\begin{aligned} & \bar{2}.7512 \times 3 \\ &= 3(-2 + 0.7512) \\ &= -6 + 2.2536 \\ &= -6 + 2 + 0.2536 \\ &= -4 + 0.2536 \\ &= \underline{\underline{\bar{4}.2536}} \end{aligned}$$

c.

$$\begin{aligned} & \bar{1}.9217 \times 3 \\ & 3(-1 + 0.9217) \\ &= -3 + 2.7651 \\ &= -3 + 2 + 0.7651 \\ &= -1 + 0.7651 \\ &= \underline{\underline{\bar{1}.7651}} \end{aligned}$$

Division of a logarithm by a whole number

Let us now consider how to divide a logarithm by a whole number. When the characteristic of a logarithm is negative the characteristic and the mantissa carry negative and positive values respectively. Therefore, it is important to divide the positive part and the negative part separately. Let us now consider some examples of this type.

Example 2

Simplify.

a. $2.5142 \div 2$ **b.** $\bar{3}.5001 \div 3$ because, $(-3 + 0.5001) \div 3$ **c.** $\bar{4}.8322 \div 2$ because, $(-4 + 0.8322) \div 2$

$2.5142 \div 2$

$= \underline{\underline{1.2571}}$

$\bar{3} \div 3 = \bar{1}$

$0.5001 \div 3 = 0.1667$

$\therefore\ \bar{3}.5001 \div 3$

$= \underline{\underline{\bar{1}.1667}}$

$\bar{4} \div 2 = \bar{2}$

$0.8322 \div 2 = 0.4161$

$\therefore\ \bar{4}.8322 \div 2$

$= \underline{\underline{\bar{2}.4161}}$

In the above example, the characteristic of the logarithm was perfectly divisible. Let us consider in the following example, how division is done when the whole part is not perfectly divisible.

Example 3

Simplify.

a. $\bar{1}.5412 \div 2$ **b.** $\bar{1}.3712 \div 3$ **c.** $\bar{3}.5112 \div 2$

a. $\bar{1}.5412 \div 2$ can be written as $(-1 + 0.5412) \div 2.$

Because the whole part, $\bar{1}$, is not perfectly divisible by 2, let us write it as $\bar{2} + 1$. Now, we can perform the division as follows

a. $\bar{1}.5412 \div 2 = (-1 + 0.5412) \div 2$

$= (-2 + 1 + 0.5412) \div 2$

$= (-2 + 1.5412) \div 2$

$= \underline{\underline{\bar{1}.7706}}$

b. $\bar{1}.3712 \div 3$

$= (-1 + 0.3712) \div 3$

$= (-3 + 2 + 0.3712) \div 3$ beacause $(-1 = -3 + 2)$

$= (\bar{3} + 2.3712) \div 3$

$= \underline{\underline{\bar{1}.7904}}$

c. $\bar{3}.5112 \div 2$

$= (-3 + 0.5112) \div 2$

$= (-4 + 1 + 0.5112) \div 2$ because $(-3 = -4 + 1)$

$= \bar{4} + 1.5112 \div 2$

$= \underline{\underline{\bar{2}.7556}}$

These types of divisions and multiplications are important when simplifying using the table of logarithms. Do the following exercise to strengthen this knowledge.

Exercise 3.5

1. Find the value.

a. $\bar{1}.5413 \times 2$ **b.** $\bar{2}.7321 \times 3$ **c.** 1.7315×3

d. 0.4882×3 **e.** $\bar{3}.5111 \times 2$ **f.** $\bar{3}.8111 \times 4$

2. Find the value.

a. $1.9412 \div 2$ **b.** $0.5512 \div 2$ **c.** $\bar{2}.4312 \div 2$

d. $\bar{3}.5412 \div 3$ **e.** $\bar{2}.4712 \div 2$ **f.** $\bar{4}.5321 \div 2$

g. $\bar{1}.5432 \div 2$ **h.** $\bar{2}.9312 \div 3$ **i.** $\bar{3}.4112 \div 2$

j. $\bar{1}.7512 \div 3$ **k.** $\bar{4}.1012 \div 3$ **l.** $\bar{5}.1421 \div 3$

3.6 Finding powers and roots of numbers using the table of logarithms

Recall that $\log_2 5^3 = 3 \log_2 5$.

This follows from the logarithmic rule $\log_a m^r = r \log_a m$.

Similarly, the logarithm of a root can be written using this rule, as follows.

(i) $\log_a \sqrt{5} = \log_a 5^{\frac{1}{2}}$ (because $\sqrt{5} = 5^{\frac{1}{2}}$)

$= \underline{\underline{\frac{1}{2} \log_a 5}}$ (using the above logarithmic rule)

(ii) $\lg \sqrt{25} = \lg 25^{\frac{1}{2}}$

$= \underline{\underline{\frac{1}{2} \lg 25}}$

The following examples consider how to extract roots and powers of a number using the table of logarithms.

Example 1

Find the value.

a. 354^2 **b.** 0.0275^3 **c.** 0.9073^4

a. **Take** $P = 354^2$.

$\lg P = \lg 354^2$

$= 2 \lg 354$

$= 2 \lg 3.54 \times 10^2$

$= 2 \times 2.5490$

$= 5.0980$

$\therefore \quad P = \text{antilog } 5.0980$

$= 1.253 \times 10^5$

$= \underline{\underline{125\,300}}$

b. **Take** $P = 0.0275^3$.

$\lg P = \lg 0.0275^3$

$= 3 \lg 0.0275$

$= 3 \times \overline{2}.4393$

$= 3 \times (-2 + 0.4393)$

$= -6 + 1.3179$

$= -6 + 1 + 0.3179$

$= -5 + 0.3179$

$= \overline{5}.3179$

$\therefore \quad P = \text{antilog } \overline{5}.3179$

$= 2.079 \times 10^{-5}$

$= \underline{\underline{0.00002079}}$

c. **Take** $P = 0.9073^4$.

$\lg P = \lg 0.9073^4$

$= 4 \lg 0.9073$

$= 4 \times \overline{1}.9577$

$= 4 \times (-1 + 0.9577)$

$= -4 + 3.8308$

$= -4 + 3 + 0.8308$

$= -1 + 0.8308$

$= \overline{1}.8308$

$\therefore P = \text{antilog } \overline{1}.8308$

$= 6.773 \times 10^{-1}$

$= \underline{\underline{0.6773}}$

Simplifying using indices

$0.9073^4 = \left(10^{\overline{1}.9577}\right)^4$

$= 10^{\overline{1}.9577 \times 4}$

$= 10^{\overline{1}.8308}$

$= 6.773 \times 10^{-1}$

$= \underline{\underline{0.6773}}$

Example 2

(i) $\sqrt{8.75}$ (ii) $\sqrt[3]{0.9371}$ (iii) $\sqrt[3]{0.0549}$

(i) Take $P = \sqrt{8.75}$.

$P = \sqrt{8.75}$ can be written as

$$P = 8.75^{\frac{1}{2}}$$
$$\lg P = \lg 8.75^{\frac{1}{2}}$$
$$= \frac{1}{2}\lg 8.75$$
$$= \frac{1}{2} \times 0.9420$$
$$= 0.4710$$
$$\therefore \quad P = \text{antilog } 0.4710$$
$$= \underline{\underline{2.958}}$$

(ii) Take $P = \sqrt[3]{0.9371}$.

$$P = 0.9371^{\frac{1}{3}}$$
$$\lg P = \lg 0.9371^{\frac{1}{3}}$$
$$= \frac{1}{3}\lg 0.9371$$
$$= \frac{1}{3} \times \bar{1}.9717$$
$$= (\bar{1}.9717) \div 3$$
$$= (-1 + 0.9717) \div 3$$
$$= (-3 + 2 + 0.9717) \div 3$$
$$= (-3 + 2.9717) \div 3$$
$$= -1 + 0.9906$$
$$= \bar{1}.9906$$
$$\therefore \quad P = \text{antilog } \bar{1}.9906$$
$$= \underline{\underline{0.9786}}$$

Simplifying using indices.

$$\sqrt[3]{0.9371} = 0.9371^{\frac{1}{3}}$$
$$= \left(10^{\bar{1}.9717}\right)^{\frac{1}{3}}$$
$$= 10^{\bar{1}.9717 \times \frac{1}{3}}$$
$$= 10^{\bar{1}.9906}$$
$$= 9.786 \times 10^{-1}$$
$$= \underline{\underline{0.9786}}$$

(iii)Take $P = \sqrt[3]{0.0549}$.

$$\begin{aligned}
\lg P &= \lg 0.0549^{\frac{1}{3}} \\
&= \frac{1}{3} \lg 0.0549 \\
&= \frac{1}{3} \times \bar{2}.7396 \\
&= (\bar{2}.7396) \div 3 \\
&= (-2 + 0.7396) \div 3 \\
&= (-3 + 1 + 0.7396) \div 3 \\
&= (-3 + 1.7396) \div 3 \\
&= -1 + 0.5799 \\
&= \bar{1}.5799 \\
\therefore P &= \text{antilog } \bar{1}.5799 \\
&= \underline{\underline{0.3801}}
\end{aligned}$$

Simplifying using indices.

$$\begin{aligned}
\sqrt[3]{0.0549} &= 0.0549^{\frac{1}{3}} \\
&= \left(10^{\bar{2}.7396}\right)^{\frac{1}{3}} \\
&= 10^{\bar{2}.7396 \times \frac{1}{3}} \\
&= 10^{\bar{1}.5799} \\
&= 3.801 \times 10^{-1} \\
&= \underline{\underline{0.3801}}
\end{aligned}$$

Now do the following exercise.

Exercise 3.6

1. Find the value using the table of logarithms.

a. $(5.97)^2$ **b.** $(27.85)^3$ **c.** $(821)^3$

d. $(0.752)^2$ **e.** $(0.9812)^3$ **f.** $(0.0593)^2$

2. Find the value using the table of logarithms.

a. $\sqrt{25.1}$ **b.** $\sqrt{947.5}$ **c.** $\sqrt{0.0714}$

d. $\sqrt[3]{0.00913}$ **e.** $\sqrt[3]{0.7519}$ **f.** $\sqrt{0.999}$

3.7 Simplification of expressions involving powers and roots using the table of logarithms

The following example demonstrates how to simplify an expression involving roots, powers, products and divisions (or some of these) using the table of logarithms.

Example 1

Simplify. Give your answer to the nearest first decimal place.

a. $\dfrac{7.543 \times 0.987^2}{\sqrt{0.875}}$ **b.** $\dfrac{\sqrt{0.4537} \times 75.4}{0.987^2}$

a. Take $P = \dfrac{7.543 \times 0.987^2}{\sqrt{0.875}}$

$$
\begin{aligned}
\text{Then } \lg P &= \lg\left(\frac{7.543 \times 0.987^2}{\sqrt{0.875}}\right) \\
&= \lg 7.543 + \lg 0.987^2 - \lg 0.875^{\frac{1}{2}} \\
&= \lg 7.543 + 2 \lg 0.987 - \frac{1}{2} \lg 0.875 \\
&= 0.8776 + 2 \times \bar{1}.9943 - \frac{1}{2} \times \bar{1}.9420 \\
&= 0.8776 + 2 \times \bar{1}.9943 - \frac{\bar{2} + 1.9420}{2} \\
&= 0.8776 + \bar{1}.9886 - (\bar{1} + 0.9710) \\
&= 0.8776 + \bar{1}.9886 - \bar{1}.9710 \\
&= 0.8662 - \bar{1}.9710 \\
&= 0.8952 \\
\therefore \quad P &= \text{antilog } 0.8952 \\
&= 7.855
\end{aligned}
$$

$$\therefore \frac{7.543 \times 0.987^2}{\sqrt{0.875}} \approx \underline{\underline{7.9}} \quad \text{(to the nearest first decimal place)}$$

This simplification can be done by using indices too as follows.

Simplifying using indices.

$$\frac{7.543 \times 0.987^2}{\sqrt{0.875}} = \frac{7.543 \times 0.987^2}{0.875^{\frac{1}{2}}}$$

$$= \frac{10^{0.8776} \times \left(10^{\bar{1}.9943}\right)^2}{\left(10^{\bar{1}.9420}\right)^{\frac{1}{2}}}$$

$$= \frac{10^{0.8776} \times 10^{\bar{1}.9886}}{10^{\bar{1}.9710}}$$

$$= \frac{10^{0.8662}}{10^{\bar{1}.9710}}$$

$$= 10^{0.8662 - \bar{1}.9710}$$

$$= 10^{0.8952}$$

$$= 7.855 \times 10^0$$

$$= 7.855$$

$$\therefore \frac{7.543 \times 0.987^2}{\sqrt{0.875}} \approx 7.9 \quad \text{(to the nearest first decimal place)}$$

b. Take $P = \dfrac{\sqrt{0.4537} \times 75.4}{0.987^2}$

$$\lg P = \lg\left(\frac{0.4537^{\frac{1}{2}} \times 75.4}{0.987^2}\right)$$

$$= \lg 0.4537^{\frac{1}{2}} + \lg 75.4 - \lg 0.987^2$$

$$= \frac{1}{2} \lg 0.4537 + \lg 75.4 - 2 \lg 0.987$$

$$= \frac{1}{2} \times \bar{1}.6568 + 1.8774 - 2 \times \bar{1}.9943$$

$$= \bar{1}.8284 + 1.8774 - \bar{1}.9886$$

$$= 1.7058 - \bar{1}.9886$$

$$= 1.7172$$

$$P = \text{antilog } 1.7172$$

$$= \underline{\underline{52.15}}$$

$$\frac{\sqrt{0.4537} \times 75.4}{0.987^2} \approx \underline{\underline{52.2}} \quad \text{(to the nearest first decimal place)}$$

Simplifying using indices is given below.

$$\frac{\sqrt{0.4537} \times 75.4}{0.987^2} = \left(\frac{0.4537^{\frac{1}{2}} \times 75.4}{0.987^2}\right)$$

$$= \frac{\left(10^{\bar{1}.6568}\right)^{\frac{1}{2}} \quad 10^{1.8774}}{\left(10^{\bar{1}.9943}\right)^2}$$

$$= \frac{10^{\bar{1}.8284} \times 10^{1.8774}}{10^{\bar{1}.9886}}$$

$$= 10^{1.7058 - \bar{1}.9886}$$

$$= 10^{1.7172}$$

$$= 52.15$$

$$\approx \underline{\underline{52.2}} \text{ (to the nearest first decimal place)}$$

Exercise 3.7

Use the table of logarithms to compute the value.

a. $\dfrac{8.765 \times \sqrt[3]{27.03}}{24.51}$ **b.** $\dfrac{\sqrt{9.18} \times 8.02^2}{9.83}$ **c.** $\dfrac{\sqrt{0.0945} \times 4.821^2}{48.15}$

d. $\dfrac{3 \times 0.752^2}{\sqrt{17.96}}$ **e.** $\dfrac{6.591 \times \sqrt[3]{0.0782}}{0.9821^2}$ **f.** $\dfrac{3.251 \times \sqrt[3]{0.0234}}{0.8915}$

3.8 Applications of logarithms

The table of logarithms can be used to do computations efficiently in many problems that involve products and divisions of numbers. Such an example is given below.

Example 1

The volume V, of a sphere of radius r is given by, $V = \frac{4}{3}\pi r^3$. By taking $\pi = 3.142$ and given that $r = 0.64$ cm, use the table of logarithms to find the volume of the sphere to the nearest first decimal place.

$$V = \frac{4}{3}\pi r^3$$
$$= \frac{4}{3} \times 3.142 \times 0.64^3$$
$$\lg V = \lg\left(\frac{4}{3} \times 3.142 \times 0.64^3\right)$$
$$= \lg 4 + \lg 3.142 + 3 \lg 0.64 - \lg 3$$
$$= 0.6021 + 0.4972 + 3 \times \bar{1}.8062 - 0.4771$$
$$= 0.6021 + 0.4972 + \bar{1}.4186 - 0.4771$$
$$= 0.5179 - 0.4771$$
$$= 0.0408$$
$$\therefore \quad V = \text{antilog } 0.0408$$
$$= 1.098$$
$$\approx 1.1 \quad \text{(to the nearest first decimal place)}$$

$\therefore$ The volume of the sphere is 1.1 cm^3.

Exercise 3.8

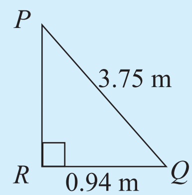
1. The mass of one cubic centimeter of iron is 7.86 g. Find the mass, to the nearest kilogram, of a cuboidal shaped iron beam, of length, width and depth respectively 5.4 m, 0.36 m and 0.22 m.

2. Find the value of g, if g is given by $g = \frac{4\pi^2 l}{T^2}$ where, $\pi = 3.142$, $l = 1.75$ and $T = 2.7$

3. A circular shaped portion of radius 0.07 m was removed from a thin circular metal sheet of radius 0.75 m.

(i) Show that the area of the remaining part is $\pi \times 0.82 \times 0.68$.
(ii) Taking π as 3.142, find the area of the remaining part using the table of logarithms.

4. The figure shows a right triangular block of land. If the dimensions of two sides are 3.75 m and 0.94 m, show that the length of PR is $\sqrt{4.69 \times 2.81}$ and find the length of PR in metres to the nearest second decimal place.



3.9 Using a calculator

Logarithms have been used for a long time to do complex numerical computations. However, its use has now been replaced to a great extent by calculators. Computations that can be done using an ordinary calculator is limited. For complex computations one needs to use a scientific calculator. The keyboard of a scientific calculator is much more complex than that of an ordinary calculator.

Evaluating powers using a calculator:

521^3 can be computed by entering $521 \times 521 \times 521$ into an ordinary calculator. However, this can be computed easily using a scientific calculator, by either using the key indicating x^n or by $\boxed{\wedge}$.

Example 1

Find the value of 275^3 using a calculator.
Show the sequence of keys that need to be activated to find 275^3.

$\boxed{2}\ \boxed{7}\ \boxed{5}\ \boxed{x^n}\ \boxed{3}\ \boxed{=}$ or $\boxed{2}\ \boxed{7}\ \boxed{5}\ \boxed{\wedge}\ \boxed{3}\ \boxed{=}$

20 796 875

Evaluating roots using a calculator:

You need to use the $\boxed{\text{shift}}$ key, when finding roots. In addition to that, you also need to activate the keys denoted by $\boxed{\sqrt[x]{}}$.

Example 1

Show the sequence of keys that need to be activated to find $\sqrt[4]{2313\,441}$ using a calculator.

$\boxed{2}\ \boxed{3}\ \boxed{1}\ \boxed{3}\ \boxed{4}\ \boxed{4}\ \boxed{1}\ \boxed{\text{shift}}\ \boxed{x^n}\ \boxed{4}\ \boxed{=}$

or

$\boxed{2}\ \boxed{3}\ \boxed{1}\ \boxed{3}\ \boxed{4}\ \boxed{4}\ \boxed{1}\ \boxed{x^{1/n}}\ \boxed{4}\ \boxed{=}$

or

$\boxed{2}\ \boxed{3}\ \boxed{1}\ \boxed{3}\ \boxed{4}\ \boxed{4}\ \boxed{1}\ \sqrt[n]{x}\ \boxed{4}\ \boxed{=}$

39

Simplifying expressions involving powers and roots using a calculator:

Show the sequence of keys that need to be activated to find the value of

$$\frac{5.21^3 \times \sqrt[3]{4.3}}{3275}$$

$\boxed{5}\ \boxed{.}\ \boxed{2}\ \boxed{1}\ \boxed{x^n}\ \boxed{3}\ \boxed{\times}\ \boxed{4}\ \boxed{.}\ \boxed{3}\ \boxed{x^{1/n}}\ \boxed{3}\ \boxed{\div}\ \boxed{3}\ \boxed{2}\ \boxed{7}\ \boxed{5}\ \boxed{=}$ 0.070219546

Exercise 3.9

1. Show the sequence of keys that need to be activated to find each of the following values.

a. 952^2 **b.** $\sqrt{475}$

c. 5.85^3 **d.** $\sqrt[3]{275.1}$

e. $375^2 \times \sqrt{52}$ **f.** $\sqrt{4229} \times 352^2$

g. $\dfrac{37^2 \times 853}{\sqrt{50}}$ **h.** $\dfrac{\sqrt{751} \times 85^2}{\sqrt[3]{36}}$

i. $\dfrac{\sqrt{1452} \times 38.75}{98.2}$ **j.** $\dfrac{\sqrt[3]{827.3} \times 5.41^2}{9.74}$

Miscellaneous Exercise

1. Simplify using the table of logarithms. Verify your answer using a calculator.

(i) $\dfrac{1}{275.2}$ (ii) $\dfrac{1}{\sqrt{982.1}}$ (iii) $\dfrac{1}{\sqrt{0.954}}$

(iv) $0.5678^{\frac{1}{3}}$ (v) $0.785^2 - 0.0072^2$ (vi) $9.84^2 + 51.2^2$

2. Find the value of

(i) $\sqrt{\dfrac{a}{b}}$

(ii) $(ab)^2$

when $a = 0.8732$ and $b = 3.168$.

3. In $A = p\left(1 + \dfrac{r}{100}\right)^n$, find the value of A, when $P = 675$, $r = 3.5$ and $n = 3$.

4. A sector with an angle of 73° subtended at the center, was removed from a thin circular sheet.

(i) What fraction of the area of the circle is the area of the sector?

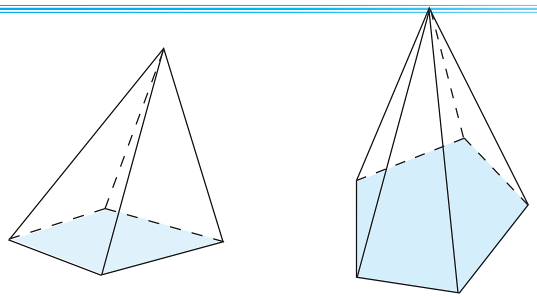
(ii) If the radius of the circle is 17.8 cm, find the area of the sector.

4 Surface area of solids

By studying this lesson you will be able to,

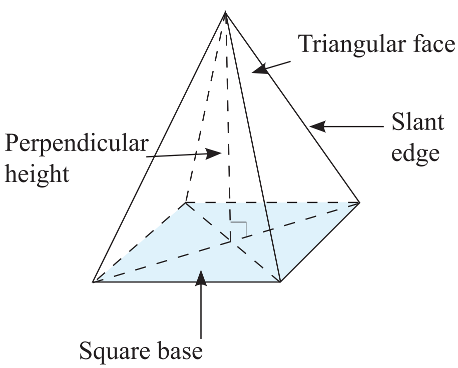
- find the surface area of a right pyramid with a square base,
- find the surface area of a right circular cone, and
- find the surface area of a sphere.

Pyramid



Carefully observe the solid objects in the above figure. Note that their faces are polygons. Of these faces, the horizontal face at the bottom is called the base. All the faces, except the base are of triangular shape. The common point of these triangular faces is called the "apex". A solid object with these properties is called a "pyramid". Note that the bases of the Pyramids shown above are respectively, the shape of a quadrilateral, a pentagon and a hexagon.

Right pyramid with a square base



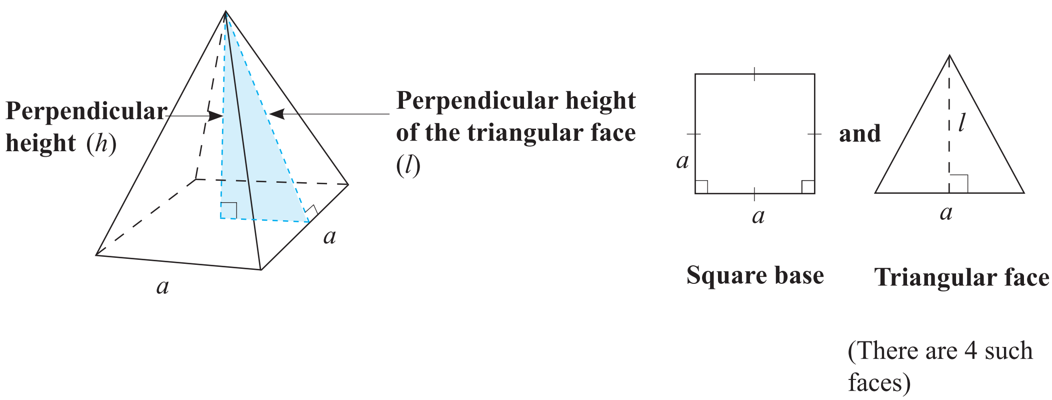
The base of the pyramid in the figure is a square. All the remaining faces are triangular in shape. If the line segment connecting the apex and the midpoint of the square base (that is the intersection point of the two diagonals) is perpendicular to the base, then such a pyramid is called a "**square based right pyramid**". The length of the line segment connecting the apex and the midpoint of the base is called the **perpendicular height** (or simply the height) of the pyramid. The edges of the triangular faces which are not common to the base are called **slant edges**. In this lesson, we will only consider finding the surface area of square based right pyramids.

Note: A tetrahedron can also be considered as a pyramid. All the faces of a tetrahedron are triangular in shape. Any one of the faces can be taken as the base. The concept of "right pyramid" can be defined even when the base is not a square. For example, we can define a right pyramid when the base of a pyramid is a regular polygon, as follows. First note that all the axes of symmetry of a regular polygon pass through a common point, which is called the centroid of the regular polygon. A pyramid, having a base which is a regular polygon, is called a right pyramid, if the line segment connecting the apex and the centroid of the base is perpendicular to the base.
If you study mathematics further, you will learn how to define the centroid, even when the base is not a regular polygon.

An important property of a square based right pyramid is that all its triangular faces are congruent to each other. Therefore, all the triangular faces have the same area. Moreover, note that each triangular face is an isosceles or an equilateral triangle, with one side a side of the square base and the other two sides equal in length.

4.1 Surface area of a square based right pyramid

To find the total surface area of a square based right pyramid we need to add the areas of the base and the four triangular faces.
Suppose the length of a side of the square base is "a" and the perpendicular height of a triangular face is "l".



Now, we can find the total surface area as follows.

$$\left.\begin{array}{c}\text{Total surface area of}\\ \text{the square based right}\\ \text{pyramid}\end{array}\right\} = \left\{\begin{array}{c}\text{Area of the}\\ \text{square base}\end{array}\right\} + 4 \times \left\{\begin{array}{c}\text{Area of a triangular}\\ \text{face}\end{array}\right\}$$

$$= a \times a + 4 \times \frac{1}{2} \times a \times l$$
$$= a^2 + 2al$$

If the total surface area is A,

$$\boxed{A = a^2 + 2al}$$

Let us now consider some solved examples on the surface area of a square based right pyramid.

Example 1

The base length of a square based right pyramid is 10 cm and the perpendicular height of a triangular face is 15 cm. Find the total surface area of the pyramid.

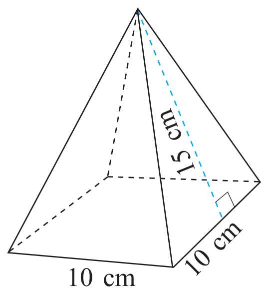
Area of the base $= 10 \times 10$
$= 100$

Area of a triangular face $= \frac{1}{2} \times 10 \times 15$
$= 75$

Area of all four triangular faces $= 75 \times 4$
$= 300$

Total surface area of the pyramid $= 100 + 300$
$= 400$

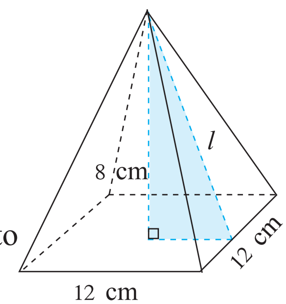
$\therefore$ Total surface area is 400 cm^2.



Example 2

Shown in the figure is a square based right pyramid of perpendicular height 8 cm and base length 12 cm. Find

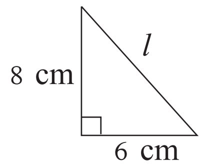
(i) the perpendicular height of a triangular face,
(ii) the area of a triangular face, and
(iii) the total surface area of the pyramid.

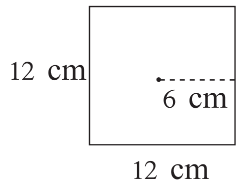


Let us take the perpendicular height of a triangular face to be l cm.
Consider the shaded triangle in the above figure.

By applying Pythagoras' theorem to this triangle,

(i) $l^2 = 8^2 + 6^2$
$= 64 + 36$
$= 100$
$\therefore\ l = \sqrt{100}$
$= 10$





$\therefore$ Perpendicular height of a triangular face is 10 cm.

(ii) Area of a triangular face $= \dfrac{1}{2} \times 12 \times 10$

$= 60$

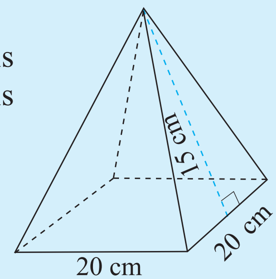
$\therefore$ Area of a triangular face is 60 cm^2.

(iii) Total surface area of the pyramid $= 12 \times 12 + 4 \times 60$
$= 144 + 240$
$= 384$

$\therefore$ Total surface area is 384 cm^2.

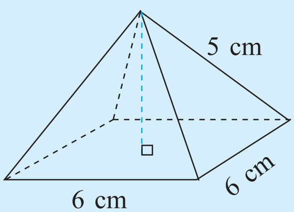
Exercise 4.1

1. The base length of a square based right pyramid is 20 cm and the perpendicular height of a triangular face is 15 cm. Find the total surface area of the pyramid.



2. In a square based right pyramid, the length of a side of the square base is 8 cm and the perpendicular height of a triangular face is 20 cm. What is the surface area of the pyramid?

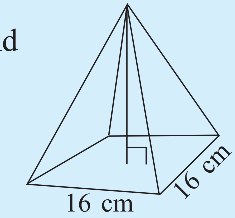
3.



The length of the slant edge of a square based right pyramid is 5 cm, and the length of a side of the base is 6 cm. Find, the total surface area of the pyramid.

4. If the length of a side of the square base of a right pyramid is 20 cm and the perpendicular height is 12 cm, find the total surface area of the pyramid.

5. The base length of a square based right pyramid is 16 cm and the perpendicular height is 6 cm. Find the



(i) perpendicular height of a triangular face.
(ii) the total surface area of the pyramid.

6. Find the total surface area of a square based right pyramid of the length of slant edge is 13 cm, and the side length of the base equal to 10 cm.

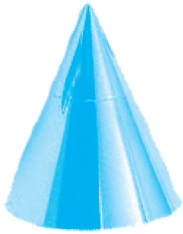
7. The surface area of a square based right pyramid is 2400 cm^2. If the length of a side of the base is 30cm, find
(i) the perpendicular distance from the apex to a side of the base, and
(ii) the height of the pyramid.

8. The area of the fabric that is used to make a tent in the shape of a square based right pyramid, is 80 m^2. Find the height of the tent, if the fabric is not used for the base of the tent and the length of a side of the base is 8cm.

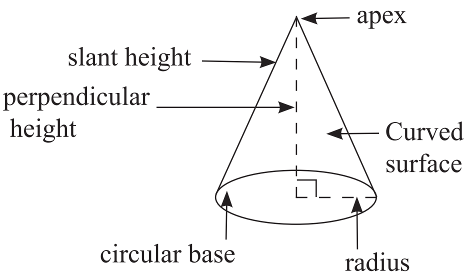
9. The height of a tent in the shape of a square based right pyramid is 4m and the perpendicular height of a triangular face is 5 m. If both the roof and the base of the tent is to be made from fabric, how much material is required?

10. It is required to construct a tent in the shape of a square based pyramid of base length 16 m and height 6 m. Find the fabric needed to construct the tent, also covering the base.

Cone



Shown above are some conical (cone shaped) objects. A cone has a **circular** plane surface and a curved surface. The circular plane surface is called the base of the cone. The point through which all the straight lines drawn on the curved surface pass through is called the "apex" of the cone.



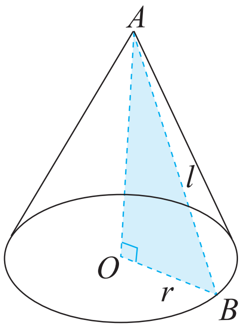
A cone is called a **right circular** cone if the line segment connecting the apex and the centre of the circular base is perpendicular to the circular base of a right circle cone. The radius of the base circle is called the radius of the cone. The length of the line segment connecting the centre of the base and the apex is the perpendicular height of a right circular cone. Moreover, any line segment connecting the apex and a point on the perimeter of the base circle is called a generator of the cone. The length of a generator is called the "slant height " of the cone.

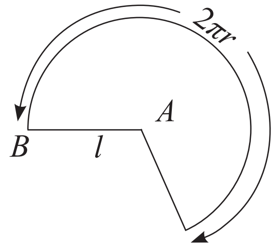
It is customary to use "r" for the radius, "h" for the height and "l" for the slant height of a cone.

4.2 Surface area of a right circular cone

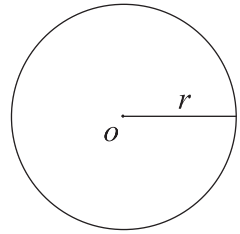
To explain a method to find the surface area of a right circular cone, consider a hollow right circular cone made from a thin sheet. Observe that the base of such a cone is a circular plane surface. Cutting open the curved surface along a generator, gives a lamina, the shape of a circular sector.

Given the radius and the slant height of a right circular cone, one can find surface area of the cone, by finding the area of the circular base and the area of the curved surface. We can use the formula πr^2 to find the area of the circular base. We can find the area of the circular sector as follows.





Curved surface



Circular base

The surface area of the curved surface is equal to the area of the sector that is obtained by cutting it open. Because the arc length of the sector is the circumference of the base circle, the arc length of the sector is equal to $2\pi r$. Also note that the radius of this sector is the slant height "l". Now, as you have learned in the lesson on the perimeter of a circular sector in Grade 10, if the angle of the sector is θ then $\frac{\theta}{360} \times 2\pi l = 2\pi r$. $\therefore\ \theta = \frac{2\pi r \times 360}{2\pi l}$ i . e, $\theta = \frac{360r}{l}$

The area of the sector with the above angle θ (as learnt in grade 10) is $\frac{\theta}{360} \times \pi l^2$.

By substituting θ from the above equation, we get $\frac{360r}{l} \times \frac{\pi l^2}{360}$. Accordingly the area of the curved surface of the cone is πrl.

Now we can add both areas, to get the total surface area of the cone.

$$\begin{aligned}\text{Total surface area of the cone} &= \left\{\begin{array}{c}\text{area of the curved}\\ \text{surface of the cone}\end{array}\right\} + \left\{\begin{array}{c}\text{area of the circular}\\ \text{base}\end{array}\right\}\\ &= \pi rl + \pi r^2\end{aligned}$$

If the total surface area is A

$$\boxed{A = \pi rl + \pi r^2}$$

Let us now consider some solved examples on the surface area of a cone. In this lesson let us take the value of π as $\frac{22}{7}$.

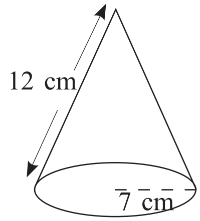
Example 1

Shown in the figure is a solid right circular cone. Its radius is 7 cm and slant height is 12 cm. Find the total surface area of the cone.

Area of the curved surface $= \pi rl$

$= \frac{22}{7} \times 7 \times 12$

$= 264$



Area of the circular base $= \pi r^2$

$= \frac{22}{7} \times 7 \times 7$

$= 154$

$\therefore$ Total surface area of the cone $= 264 + 154$

$= 418$

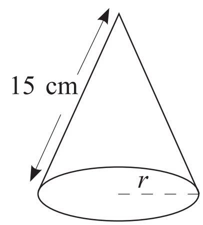
Total surface area of the cone is 418 cm^2.

Example 2

The circumference of the base of a right circular cone is 88 cm and its slant height is 15 cm. Find the area of the curved surface.

Circumference of the circular base = 88 cm

Let us take the radius as r cm.



Then, $2\pi r = 88$

$$2 \times \frac{22}{7} \times r = 88$$

$$r = \frac{88 \times 7}{2 \times 22}$$

$$r = 14$$

Surface area of the curved surface $= \pi rl$

$= \frac{22}{7} \times 14 \times 15$

$= 660$

$\therefore$ Surface area of the curved surface of the cone is 660 cm^2.

Example 3

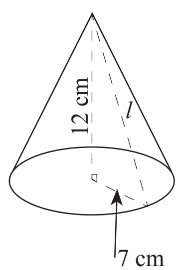
Find,

(i) the slant height,

(ii) the area of the curved surface, and

(iii) the total surface area of the cone,

accurate up to one decimal place, of a cone of radius 7 cm and perpendicular height 12 cm.



Take the slant height of the cone to be l cm.

According to Pythagoras' theorem,

$$
\begin{aligned}
\text{(i)}\quad l^2 &= 7^2 + 12^2 \\
&= 49 + 144 \\
&= 193 \\
l &= \sqrt{193} \\
&= 13.8 \quad \text{(Use the division method to find the square root)}
\end{aligned}
$$

$\therefore$ The slant height of the cone is 13.8 cm.

(ii) The area of the curved surface $= \pi r l$

$$
\begin{aligned}
&= \frac{22}{7} \times 7 \times 13.8 \\
&= 303.6
\end{aligned}
$$

$\therefore$ The area of the curved surface is 303.6 cm^2.

(iii) Area of the circular base $= \pi r^2$

$$
\begin{aligned}
&= \frac{22}{7} \times 7 \times 7 \\
&= 154
\end{aligned}
$$

Total surface area of the cone $= 303.6 + 154$

$$= 457.6$$

$\therefore$ Total surface area of the cone is 457.6 cm^2.

Exercise 4.2

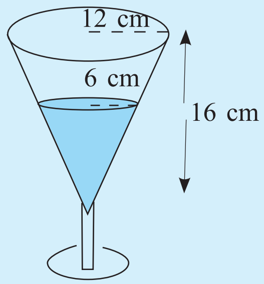
1. Find the area of the curved surface of a right circular cone of base radius 14 cm and slant height 20 cm.

2. Of a right circular solid cone, the radius of the base is 7 cm and the height is 24 cm. Find,
 (i) the slant height, and
 (ii) the area of the curved surface.

3. If the slant height, of a conical shaped sand pile with a base circumference 44 cm, is 20 cm, find
 (i) the radius of the base, and
 (ii) the area of the curved surface.

4. Find the total surface area of a right circular cone of base radius 10.5 cm and slant height 15 cm.

5. The slant height of a conical shaped solid object is 14 cm. If the area of the curved surface is 396 cm^2, find
(i) the radius of the cone, and
(ii) the perpendicular height of the cone.

6. 

Shown in the picture, is a thin glass container in the shape of a cone filled with a juice to half its height. The radius of the glass is 12 cm and its height is 16 cm. Find the area of the region on the glass surface that is in contact with the juice.

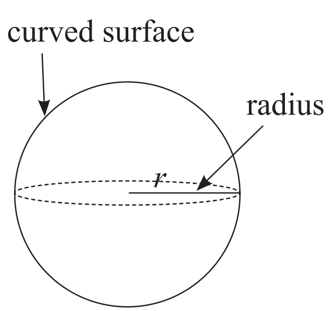
Sphere


shot put


tennis ball


Foot ball

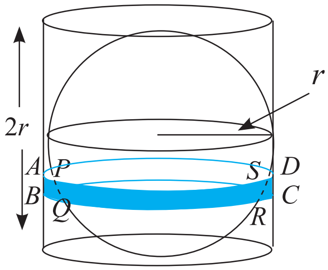
There is no doubt that you know what properties a sphere has. In mathematics, a sphere is defined as the set of points in three dimensional space that lies at a constant distance from a fixed point. The fixed point is called the centre of the sphere and the constant distance from the centre to a point on the sphere is called the radius of the sphere. A sphere has only one curved surface, and has no edges or vertices.



We will usually use "r" to indicate the radius of the sphere.

4.3 Surface area of a sphere

A cylinder with the same radius as the sphere and height equal to the diameter of the sphere is called the **circumscribing cylinder** of the sphere. The sphere will tightly fit in the circumscribing cylinder of the sphere.



The following fact regarding the surface area of a sphere and its circumscribing cylinder was observed by the Greek mathematician Archimedes, who lived around 225 B.C.

When the sphere is inside the circumscribing cylinder, any two planes parallel to the flat circular surfaces of the cylinder will bound equal surface areas on the curved surfaces of the sphere and cylinder.

For example, in the above figure, the area of the curved surface $PQRS$ on the sphere is the same as the area of the curved surface $ABCD$ on the cylinder.

Now, if we apply this fact to the entire cylinder, we see that the surface area of the sphere is equal to the surface area of the curved surface of the cylinder. We can use the formulae $2\pi rh$ to find the surface area of the curved surface of the circumscribing cylinder.

Area of the curved surface of the circumscribing cylinder $= 2\pi r \times 2r$

$= 4\pi r^2$

Therefore, the surface area of the sphere $= 4\pi r^2$

If the surface area is A,

$$\boxed{A = 4\pi r^2}$$

Example 1

Find the surface area of a sphere of radius 7 cm.

Surface area of the sphere $= 4\pi r^2$

$= 4 \times \frac{22}{7} \times 7 \times 7$

$= 616$

$\therefore$ surface area of the sphere is 616 cm^2.

Example 2

The surface area of a sphere is $1386\ \text{cm}^2$. Find the radius of the sphere.

Let r be the radius.

Then, $4\pi r^2 = 1386$

$$4 \times \frac{22}{7} \times r^2 = 1386$$

$$r^2 = \frac{1386 \times 7}{4 \times 22}$$

$$= \frac{441}{4}$$

$$r = \sqrt{\frac{441}{4}}$$

$$= \frac{21}{2}$$

$$= 10.5$$

$\therefore$ radius of the sphere is 10.5 cm.

Exercise 4.3

1. Find the surface area of a sphere of radius 3.5 cm.

2. Find the surface area of a sphere of radius 14 cm.

3. Find the radius of a sphere of surface area $5544\ \text{cm}^2$.

4. Find the (external) surface area of a hollow hemisphere of radius 7 cm.

5. Find the surface area of a solid sphere of diameter 0.5 cm.

6. Find the radius of a solid hemisphere with a surface area of $1386\ \text{cm}^2$.

Summary

- The surface area A of a square based right pyramid, of base length "a" and height "l" is

  $A = a^2 + 2al$

- The surface area A, of a right circular cone of radius r and slant height l is
  $A = \pi rl + \pi r^2$

- The surface area A of a sphere of radius r is

  $A = 4\pi r^2$.

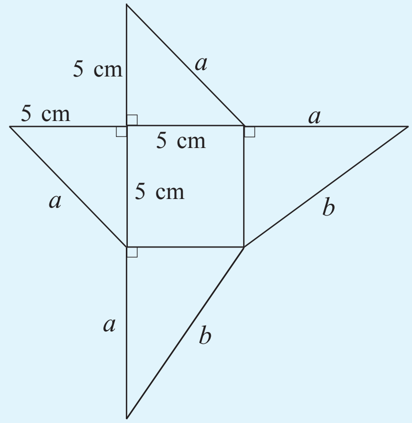
Mixed Exercise

1. Shown below is a net used to make a pyramid.

(i) Find the lengths indicated by a and b.

(ii) Give reasons as to why the resulting pyramid is not a right pyramid.

(iii) Find the total surface area of the pyramid.

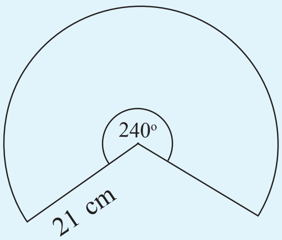


2. A right circular cone was made using a lamina in the shape of the sector shown in the figure.

A circular lamina of the same radius is fixed to the base of the cone.

(i) Find the radius of the cone.

(ii) Find the total surface area of the cone.

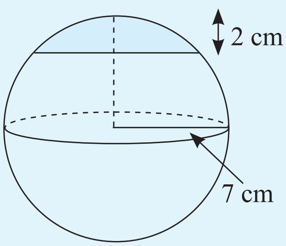


3. The ratio between the slant height and the perpendicular height of a cone is 5 : 4. The radius of the base of the cone is 6 cm.

(i) Find the slant height of the cone.

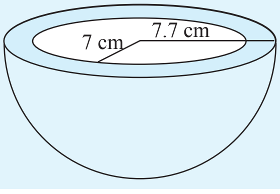
(ii) Find the surface area of the curved surface of the cone.

4.



On a sphere of radius 7 cm, paint was applied from the top downwards, a perpendicular distance of 2 cm. Find the area of the painted region. (Hint: make use of knowledge on the circumscribing cylinder)

5.



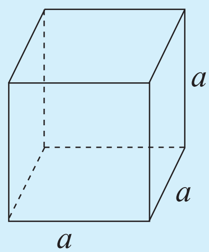
The internal radius of a hemispherical clay pot is 7 cm and the external radius is 7.7 cm. Find the total surface area of the pot.

5 Volume of solids

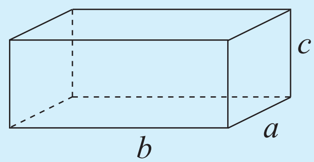
By studying this lesson you will be able to
- compute the volume of a square based right pyramid, right circular cone and a sphere.

Review Exercise

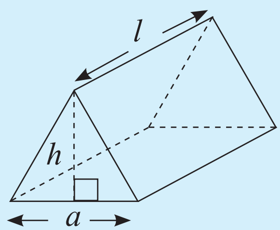
1. Shown below are figures of some solid objects that you have studied before. Complete the given table by recalling how their volume was computed.



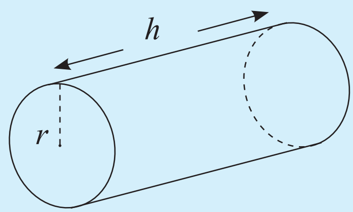
Cube



Cuboid



Triangular prism



Cylinder

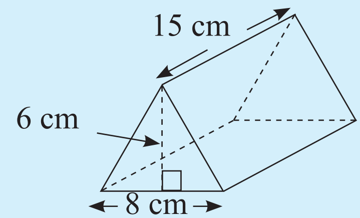
Object	Cross-sectional area	Volume
Cube		
Cuboid		
Triangular Prism		
Cylinder		

2. Find the volume of a cube of side length 10 cm.

3. Find the volume of a cuboid of length 15 cm, width 10 cm and height 8 cm.

4. Find the volume of a cylinder of radius 7 cm and height 20 cm.

5. Find the volume of the prism shown in the figure.

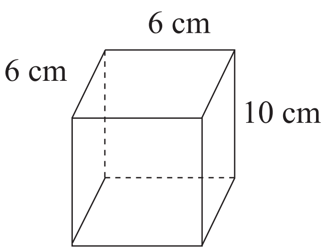


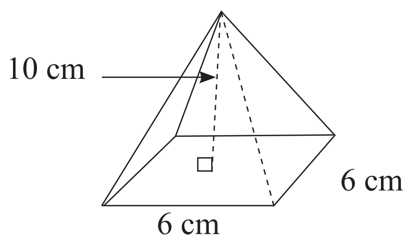
5.1 Volume of a square based right pyramid

Let us do the following activity to construct a formula to find the volume of a square based right pyramid.

Activity

Use a piece of cardboard to construct the hollow cuboid and the hollow pyramid, shown in the following figure. The cuboid has a square base with 6 cm sides and is of height 10 cm. It does not have a top. The right pyramid has a square base, again with 6 cm sides, and the height of the pyramid is also 10 cm. Do not include a base, so that you can fill it with sand.





Completely fill the pyramid with sand and empty it into the cuboid. Find how many times you need to do this to fill the cuboid.

You would have observed in the above activity, that filling the pyramid completely with sand and emptying it into the cuboid thrice, will completely fill the cuboid without any overflow.

Let us consider a square based cuboid with side length a and height h, and a square based right pyramid of side length a and height h.

According to the activity,

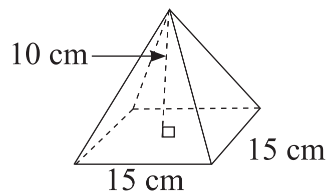
$$\text{Volume of the Pyramid} \times 3 = \text{Volume of the Cuboid}$$

$$\begin{aligned}
\therefore \quad \text{Volume of the Pyramid} &= \frac{1}{3} \times \text{Volume of the Cuboid} \\
&= \frac{1}{3} \times \text{Area of the Base} \times \text{Perpendicular Height} \\
&= \frac{1}{3} \times (a \times a) \times h \\
&= \frac{1}{3} a^2 h
\end{aligned}$$

$$\boxed{\text{Volume of the Pyramid} = \frac{1}{3} a^2 h}$$

Example 1

Find the volume of a square based right pyramid, of height 10 cm and base length 15 cm.



$$\begin{aligned}
\text{Volume of the Pyramid} &= \frac{1}{3} a^2 h \\
&= \frac{1}{3} \times 15 \times 15 \times 10 \\
&= 750
\end{aligned}$$

$\therefore$ Volume of the Pyramid is 750 cm^3.

Example 2

The volume of a pyramid with a square base is 400 cm^3. Find the length of a side of the base, if its height is 12 cm.

Let us take the length of a side of the base to be "a" cm.

$$\text{Volume of the Pyramid} = \frac{1}{3} a^2 h$$

$$\begin{aligned}
\therefore \quad \frac{1}{3} a^2 h &= 400 \\
\therefore \quad \frac{1}{3} a^2 \times 12 &= 400 \\
\therefore \quad 4a^2 &= 400 \\
\therefore \quad a^2 &= 100 \\
&= 10^2 \\
\therefore \quad a &= 10
\end{aligned}$$

$\therefore$ Length of a side of the base is 10 cm.

Exercise 5.1

1. The height of a pyramid with a square base is 9 cm. The length of a side of the base is 5 cm. Find the volume of the pyramid.

2. A square pyramid of height 10 cm has a base of area 36 cm^2. Find the volume of the pyramid.

3. The volume of a square pyramid of height 12 cm is 256 cm^3 . Find the length of a side of the base.

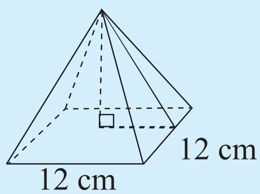
4. The height of a square pyramid is 5 cm. Its volume is 60 cm^3 . Find the area of its base.

5. The volume of a square pyramid is 216 cm^3 . The length of a side of its base is 9 cm. Find the height of the pyramid.

6. The area of the base of a square pyramid is 16 cm^2 and its volume is 80 cm^3. Find the height of the pyramid.

7. The side length of the base of a square pyramid is 12 cm and the slant height is 10 cm. Find

 (i) the height, and
 (ii) the volume
 of the pyramid.



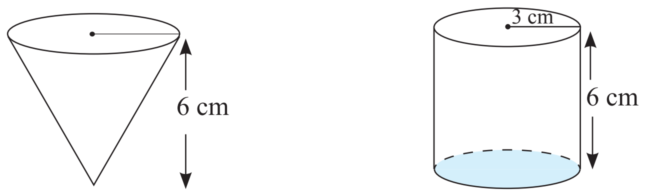
8. The side length of the base of a square pyramid is 10 cm and the slant height is 13 cm. Find
 (i) the height, and
 (ii) the volume
 of the pyramid.

5.2 Volume of a right circular cone

Let us consider constructing a formula for the volume of a right circular cone. Do the following activity using a right circular cone and a right circular cylinder.

Activity

As shown in the figure, using a cardboard, construct a cone without a base, and a cylinder with a base but without a lid, of equal radii and equal height.



Fill the cone completely with sand and empty it into the cylinder. Find how many times you need to do this in order to completely fill the cylinder.

You will be able to observe that, pouring three times from the cone will completely fill the cylinder without any overflow. According to this observation,

$$\text{Volume of the cone} \times 3 = \text{Volume of the cylinder}$$

$$\text{Volume of the cone} = \frac{1}{3} \times \text{Volume of the cylinder}$$

You have learned in a previous lesson that the volume of a cylinder, of radius r and height h, is given by $\pi r^2 h$. Therefore, the volume of a cone of radius r and height h is given by $\frac{1}{3}\pi r^2 h$.

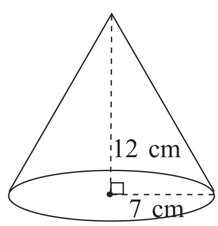
$$\boxed{\text{Volume of the cone} = \frac{1}{3}\pi r^2 h}$$

The value of π is taken as $\frac{22}{7}$ in this lesson.

Example 1

Find the volume of a cone of radius 7 cm and height 12 cm.

$$\begin{aligned}\text{Volume of the cone} &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 12 \\ &= 616\end{aligned}$$



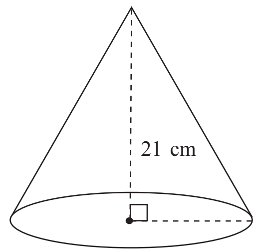
$\therefore$ volume of the cone is $616\ \text{cm}^3$.

Example 2

The circumference of the base of a cone is 44 cm. Its perpendicular height is 21 cm. Find the volume of the cone.

Circumference of the base = 44 cm
Let us take the radius as r centimetres

$$\therefore \; 2\pi r = 44$$
$$2 \times \frac{22}{7} \times r = 44$$
$$\therefore \; r = \frac{44 \times 7}{2 \times 22}$$
$$= 7$$



$\therefore$ radius of the cone is 7 cm.

$$\text{Volume of the cone} = \frac{1}{3}\pi r^2 h$$
$$= \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 21$$
$$= 1078$$

$\therefore$ volume of the cone is 1078 cm^3.

Example 3

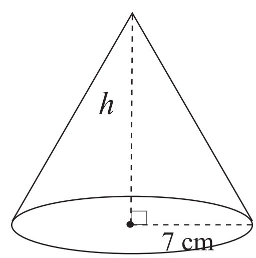
Find,

(i) the height
(ii) the volume

of a cone of radius 7 cm and slant height 25 cm.

Let us indicate the height of the cone by h centimetres. Let us apply Pythagoras' Theorem to the indicated triangle of the cone.

(i)
$$h^2 + 7^2 = 25^2$$
$$h^2 + 49 = 625$$
$$h^2 = 625 - 49$$
$$h = \sqrt{576}$$
$$h = 24$$



$\therefore$ height of the cone is 24 cm.

(ii) volume of the cone $= \frac{1}{3}\pi r^2 h$

$$= \frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 24$$

$$= 1232$$

$\therefore$ volume of the cone is 1232 cm^3.

Example 4

Find the perpendicular height of a cone of radius 3.5 cm and volume 154 cm^3.

Let us indicate the perpendicular height of the cone by h centimetres.

Volume of the cone $= \frac{1}{3}\pi r^2 h$

$$\therefore\ 154 = \frac{1}{3} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times h \qquad \left(\text{because } 3.5 = \frac{7}{2}\right)$$

$$\therefore\ h = \frac{154 \times 3 \times 7 \times 2 \times 2}{22 \times 7 \times 7}$$

$$= 12$$

$\therefore$ perpendicular height of the cone is 12 cm.

Exercise 5.2

1. Find the volume of a cone of radius 7 cm and height 12 cm.

2. Find the volume of a cone of diameter 21 cm and height 25 cm.

3. Find the volume of a cone of slant height 13 cm and base radius 5 cm.

4. Find the volume of a cone of diameter 12 cm and slant height 10 cm.

5. If the height of a cone, of volume 616 cm^3 is 12 cm, find the radius of the cone.

6. The volume of a cone is 6468 cm^3 and its height is 14 cm. Find the diameter of the cone.

7. The circumference of the base of a right cone is 44 cm and its slant height is 25 cm. Compute the following.

(i) Radius of the base

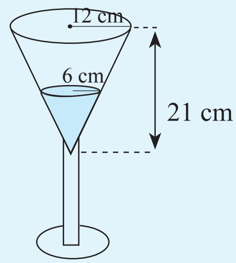
(ii) Height

(iii) Volume

8. The circumference of the base of a conical shaped container is 88 cm and its perpendicular height is 12 cm. Find the volume of the container.

9. How many cones of radius 7 cm and height 15 cm can be made by melting a solid metal cylinder of base radius 14 cm and height 30 cm?

10. A container is of the form of an inverted right circular cone. The radius of the cone is 12 cm and its height is 21 cm. If the container is filled with water to half its height, how much more water is required to fill it completely?

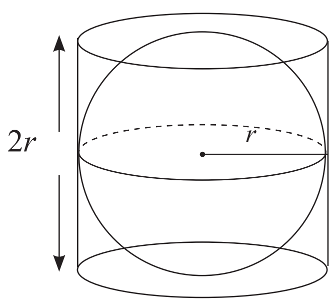


5.3 Volume of a sphere

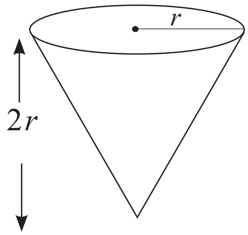
Recall the circumscribing cylinder of a sphere, that we leaned about in the lesson on the surface area of a sphere. Archimedes, the Greek mathematician who explained the surface area of a sphere in terms of the circumscribing cylinder, also explained the volume of a sphere in terms of the volume of the circumscribing cylinder. Let us do the following activity to understand his finding.

Activity

Find a small sphere of radius 3 cm. Using a piece of cardboard, create a cylinder with the same radius as that of the sphere and height equal to the diameter of the sphere. Do not close the two ends of the cylinder. Now carefully insert the sphere into the cylinder.



Take the cylinder, with the sphere inside it. Fill the top part of the cylinder, that is not ocupied by the sphere with sand. Cover top with a piece of cardboad and flip it over and fill the remaining part that is not occupied by the sphere also with sand. Put the total amount of sand that you used to fill the remaining volume of the cylinder after inserting the sphere into the cone that you made earlier. Note that the volume of sand fills the cone completely.



Let us formulate what you have observed in this activity.

Volume of the Circumscribing Cylinder = Volume of the Sphere + Volume of the Cone

Therefore, we can find the volume of the sphere by subtracting the volume of the cone from the volume of the circumscribing cylinder.

Volume of the Sphere = Volume of the Circumscribing Cylinder – Volume of the Cone

$$= \pi r^2 h - \frac{1}{3} \times \pi r^2 h$$

$$= \frac{2}{3} \pi r^2 h$$

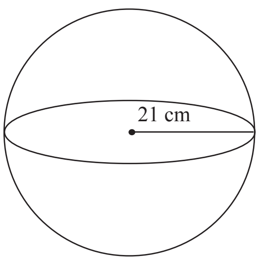
$$= \frac{2}{3} \pi r^2 \times 2r \quad \text{(Because } h = 2r\text{)}$$

$$= \frac{4}{3} \pi r^3$$

$$\boxed{\text{Volume of the Sphere } = \frac{4}{3} \pi r^3}$$

Example 1

Find the volume of a sphere of radius 21 cm.



Volume of the Sphere $= \frac{4}{3} \pi r^3$

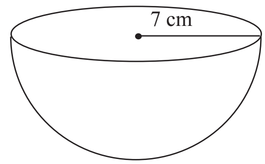
$$= \frac{4}{3} \times \frac{22}{7} \times 21 \times 21 \times 21$$

$$= 38\,808$$

$\therefore$ Volume of the sphere is $38\,808 \text{ cm}^3$.

Example 2

Find the volume of a solid hemisphere of radius 7 cm.



$$\text{Volume of the hemisphere} = \frac{1}{2} \times \frac{4}{3}\pi r^3$$

$$= \frac{1}{2} \times \frac{4}{3} \times \frac{22}{7} \times 7 \times 7 \times 7$$

$$= 718.67$$

$\therefore$ Volume of the hemisphere is approximately $718.67\ \text{cm}^3$.

Example 3

Find the radius of a small spherical marble, of volume $113\frac{1}{7}\ \text{cm}^3$.
Let us take the radius as r centimetres.

$$\text{Volume of the sphere} = \frac{4}{3}\pi r^3$$

$$\therefore \quad \frac{4}{3}\pi r^3 = 113\frac{1}{7}$$

$$\therefore \quad r^3 = \frac{792}{7} \times \frac{3}{4} \times \frac{7}{22}$$

$$= 27$$

$$= 3^3$$

$$\therefore \quad r = 3$$

$\therefore$ Radius of the sphere is 3 cm.

Exercise 5.3

1. Find the volume of a sphere of radius 7 centimeters.

2. Show that the volume of a sphere of diameter 9 centimeters is $381\frac{6}{7}\ \text{cm}^3$.

3. Find the volume of a spherical celestial body of radius 2.1 kilometers.

4. Find the volume of a hemisphere of radius 10.5 centimeters.

5. The volume of a sphere is $11492\frac{2}{3}$ cubic centimeters. Find the radius of the sphere.

6. Find the radius of the metal ball that is made by using all the metal obtained by melting 8 metal balls, each of radius 7 cm.

7. Show that 32 metal balls of radius 3 cm can be made by melting a solid metal hemisphere of radius 12 cm.

Summary

- The volume V of a square based right pyramid, of base length "a" and height "h" is

  $$V = \frac{1}{3}a^2h.$$

- The volume V, of a right circular cone of radius r and height h is

  $$V = \frac{1}{3}\pi r^2h.$$

- The volume V of a sphere of radius r is

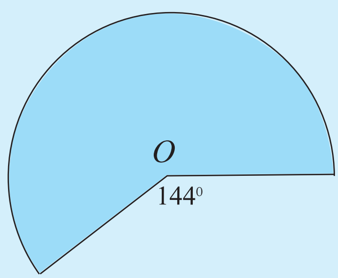
  $$V = \frac{4}{3}\pi r^3.$$

Miscellaneous Exercise

1. Solid metal spheres of radius 3 cm were made by melting a metal block. The metal block had a square crossection of 12 cm sides. The length of the metal block was 22 cm. How many metal spheres were made?

2. A solid metal sphere of radius 3.5 cm was melted and casted into a circular cone of the same radius. Find the height of the cone, assuming there was no waste of the metal in the molding process.

3. A cone of slant height r and apex O was constructed using a metal lamina in the shape of the sector shown in the figure. The centre and the radius of the sector are O and r respectively. n pieces of ice in the shape of spheres of radius a are placed in this cone. (the cone is held inverted) If the cone is filled completely with water when all the ice melts show that $125na^3 = 9r^3$.



6

Binomial Expressions

By studying this lesson, you will be able to

expand the cube (third power) of a binomial expression.

You have learned in earlier lessons that, for a binomial expression of the form $x + y$, its square is denoted by $(x + y)^2$, and that what it means is $(x + y)\,(x + y)$, and that when the product is expanded, the expression $x^2 + 2xy + y^2$ is obtained. Moreover, recall that $x^2 - 2xy + y^2$ is obtained when $(x - y)^2$ is expanded.

Do the following exercise to recall what you have learned about the expansion of squares of binomial expressions.

Review Exercise

1. Fill in the blanks.

a. $(a + b)^2 = a^2 + 2ab + \ldots..$

b. $(a - b)^2 = \ldots. - 2ab + b^2$

c. $(x + 2)^2 = x^2 + 4x + \ldots..$

d. $(y + 3)^2 = y^2 + \ldots.. + 9$

e. $(a - 5)^2 = \ldots.. - 10a + 25$

f. $(b - 1)^2 = b^2 \;\ldots.. + \;\ldots..$

g. $(4 + x)^2 = 16 + \ldots.. \;\ldots..$

h. $(7 - t)^2 = 49 \;\ldots.. + t^2$

i. $(2x + 1)^2 = 4x^2 \;\ldots.. + 1$

j. $(3b - 2)^2 = \ldots\ldots - 12b \;\ldots\ldots$

2. Expand.

a. $(2m + 3)^2$ **b.** $(3x - 1)^2$ **c.** $(5+2x)^2$

d. $(2a + 3b)^2$ **e.** $(3m - 2n)^2$ **f.** $(2x + 5y)^2$

3. Evaluate the following squares, by writing each as a square of a binomial expression.

a. 32^2 **b.** 103^2 **c.** 18^2 **d.** 99^2

6.1 Cube of a binomial expression

The cube of the binomial expression $a + b$, is $(a + b)^3$. That is, the third power of $(a + b)$. Note that this is the same as multiplying $(a + b)^2$ again by $(a + b)$.

Carefully observe how the following expressions, involving a power of 3, are written.

$$3^3 = 3 \times 3^2 = 3 \times 3 \times 3 = 27$$
$$x^3 = x \times x^2 = x \times x \times x$$
$$(2x)^3 = (2x) \times (2x)^2 = (2x) \times (2x) \times (2x) = 8x^3$$

In a similar way, we can write

$$(x + 1)^3 = (x + 1)(x + 1)^2 \quad = (x + 1)\,(x + 1)\,(x + 1)$$
$$(a - 2)^3 = (a - 2)(a - 2)^2 \quad = (a - 2)\,(a - 2)\,(a - 2)$$
$$(3 + m)^3 = (3 + m)(3 + m)^2 = (3 + m)\,(3 + m)\,(3 + m)$$

The cube of a binomial expression can be expanded in a way similar to how the square of a binomial expression was expanded. It is illustrated in the following example.

Example 1

$$\begin{aligned}(x + y)^3 &= (x + y)\,(x + y)^2 \\ &= (x + y)\quad(x^2 + 2xy + y^2) \\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3 \\ &= \underline{\underline{x^3 + 3x^2y + 3xy^2 + y^3}}\end{aligned}$$

Accordingly, let us remember the following pattern as a formula for the expansion of the cube of the binomial expression $(x + y)$.

$$\boxed{(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3}$$

cube of the first term

three times the product of the square of the first term and the second term.

three times the product of the first term and the square of the second term

cube of the second term

According to this, we can write

$$(m + n)^3 = m^3 + 3m^2n + 3mn^2 + n^3$$

Similarly, we can write $(a+2)^3 = a^3 + 3a^2 \times 2 + 3a \times 2^2 + 2^3$, and this can be further simplified as ,

$a^3 + 6a^2 + 12a + 8$

Now let us consider how the expansion of $(x-y)^3$ is obtained by taking products.

$$
\begin{aligned}
(x-y)^3 &= (x-y)\,(x-y)^2 \\
&= (x-y)\quad(x^2 - 2xy + y^2) \\
&= x^3 - 2x^2y + xy^2 - x^2y + 2xy^2 - y^3 \\
&= \underline{\underline{x^3 - 3x^2y + 3xy^2 - y^3}}
\end{aligned}
$$

Now, let us consider how we can obtain the expansion of $(x-y)^3$, using another method.

First, note that we can write $x - y$ as $x + (-y)$. Therefore, we can treat $(x-y)^3$ as an expression of the initial form, by writing it as $\{x + (-y)\}^3$. Let us now consider the expansion of this cube.

$$
\begin{aligned}
(x-y)^3 = \{x + (-y)\}^3 &= x^3 + 3 \times x^2 \times (-y) + 3 \times x \times (-y)^2 + (-y)^3 \\
&= \underline{\underline{x^3 - 3x^2y + 3xy^2 - y^3}}
\end{aligned}
$$

Note that we have used the properties $(-y)^2 = y^2$ and $(-y)^3 = -y^3$ in the above simplification.

According to this, we can also write

$$
\begin{aligned}
(m-n)^3 &= m^3 - 3m^2n + 3mn^2 - n^3 \\
(p-q)^3 &= p^3 - 3p^2q + 3pq^2 - q^3
\end{aligned}
$$

Either method can be used to obtain the expansion of $(x-y)^3$. You may use any method which is easy for you.

Let us now consider how the cube of a binomial expression, involving numbers as well, is expanded.

Example 2

$$
\begin{aligned}
(x+5)^3 &= x^3 + 3 \times x^2 \times 5 + 3 \times x \times 5^2 + 5^3 \\
&= \underline{\underline{x^3 + 15x^2 + 75x + 125}}
\end{aligned}
$$

Example 3

$$
\begin{aligned}
(1+x)^3 &= 1^3 + 3 \times 1^2 \times x + 3 \times 1 \times x^2 + x^3 \\
&= \underline{\underline{1 + 3x + 3x^2 + x^3}}
\end{aligned}
$$

Example 4

$$(y-4)^3 = y^3 + 3 \times y^2 \times (-4) + 3 \times y \times (-4)^2 + (-4)^3$$
$$= \underline{\underline{y^3 - 12y^2 + 48y - 64}}$$

or

$$(y-4)^3 = y^3 - 3 \times y^2 \times 4 + 3 \times y \times 4^2 - 4^3$$
$$= \underline{\underline{y^3 - 12y^2 + 48y - 64}}$$

Example 5

$$(5-a)^3 = 5^3 + 3 \times 5^2 \times (-a) + 3 \times 5 \times (-a)^2 + (-a)^3$$
$$= \underline{\underline{125 - 75a + 15a^2 - a^3}}$$

or

$$(5-a)^3 = 5^3 - 3 \times 5^2 \times a + 3 \times 5 \times a^2 - a^3$$
$$= \underline{\underline{125 - 75a + 15a^2 - a^3}}$$

Example 6

$$(-2+a)^3 = (-2)^3 + 3 \times (-2)^2 \times a + 3 \times (-2) \times a^2 + a^3$$
$$= \underline{\underline{-8 + 12a - 6a^2 + a^3}}$$

Example 7

$$(-3-b)^3 = (-3)^3 + 3 \times (-3)^2 \times (-b) + 3 \times (-3) \times (-b)^2 + (-b)^3$$
$$= \underline{\underline{-27 - 27b - 9b^2 - b^3}}$$

or

$$\left[-1\,(3+b)\right]^3 = (-1)^3\ (3+b)^3$$
$$= -1\,(3^3 + 3 \times 3^2 \times b + 3 \times 3 \times b^2 + b^3)$$
$$= -1\,(27 + 27b + 9b^2 + b^3)$$
$$= \underline{\underline{-27 - 27b - 9b^2 - b^3}}$$

Example 8

Write the expansion of $(x - 3)^3$ and verify that $(4 - 3)^3 = 4^3 - 3^2 \times 4^2 + 3^3 \times 4 - 3^3$

$(x - 3)^3 = x^3 - 3^2 \times x^2 + 3^3 \times x - 3^3$

Substituting $x = 4$

Left s. $= (4 - 3)^3$
$= 1$

Right s. $= 4^3 - 3^2 \times 4^2 + 3^3 \times 4 - 3^3$
$= 1$

Left s. $=$ Right s.

Therefore $(4 - 3)^3 = 4^3 - 3^2 \times 4^2 + 3^3 \times 4 - 3^3$

Exercise 6.1

1. Fill in the blanks using suitable algebraic terms, symbols (+ or –) or numbers.

a. $(x + 3)^3 = x^3 + 3 \times x^2 \times 3 + 3 \times x \times 3^2 + 3^3 = x^3 + \square + \square + 27$

b. $(y + 2)^3 = y^3 + 3 \times \square \times \square + 3 \times \square \times \square + 2^3 = y^3 + 6y^2 + \square + \square$

c. $(a - 5)^3 = a^3 + 3 \times a^2 \times (-5) + 3 \times a \times (-5)^2 + (-5)^3 = a^3 - \square + \square - 125$

d. $(3 + t)^3 = \square + 3 \times \square \times \square + 3 \times \square \times \square + \square = \square + 27t + \square + t^3$

e. $(x - 2)^3 = x^3 \square\, 3 \times \square \times \square + 3 \times \square \times \square + (-2)^3 = x^3 \square \square + 12x - \square$

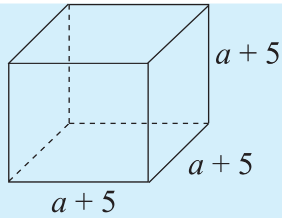
2. Expand.

a. $(m + 2)^3$	**b.** $(x + 4)^3$	**c.** $(b - 2)^3$	**d.** $(t - 10)^3$
e. $(5 + p)^3$	**f.** $(6 + k)^3$	**g.** $(1 + b)^3$	**h.** $(4 - x)^3$
i. $(2 - p)^3$	**j.** $(9 - t)^3$	**k.** $(-m + 3)^3$	**l.** $(-5 - y)^3$
m. $(ab + c)^3$	**n.** $(2x + 3y)^3$	**o.** $(3x + 4y)^3$	**p.** $(2a - 5b)^3$

3. Write as a cube of a binomial expression.

a. $a^3 + 3a^2b + 3ab^2 + b^3$	**b.** $c^3 - 3c^2d + 3cd^2 - d^3$
c. $x^3 + 6x^2 + 12x + 8$	**d.** $y^3 - 18y^2 + 108y - 216$
e. $1 + 3x + 3x^2 + x^3$	**f.** $64 - 48x + 12x^2 - x^3$

4. Shown in the diagram is a cube with the length of each side $(a + 5)$ units. Write an expression for the volume of the cube and expand it.



5. Expand $(x + 3)^3$, and verify the result for the following cases.

(i) $x = 2$
(ii) $x = 4$

.

6. Use the knowledge on cubes of binomial expressions to evaluate the following numerical expressions.

(i) $64 - 3 \times 16 \times 3 + 3 \times 4 \times 9 - 27$
(ii) $216 - 3 \times 36 \times 5 + 3 \times 6 \times 25 - 125$

7. Find the value of each of the following, by writing each as a cube of a binomial expression.

a. 21^3 **b.** 102^3 **c**. 17^3 **d**. 98^3

8. Find the volume of a cube, with each side $2a - 5$ cm, in terms of a.

9. Write $x^3 - 3x^2y + 3xy^2 - y^3$ as a cube and use it to find the value of $25^3 - 3\times 25^2\times 23 + 3 \times 25 \times 23^2 - 23^3$.

7 Algebraic Fractions

By studying this lesson, you will understand

how algebraic fractions are multiplied and divided.

Do the following exercise to revise what you have learned before on adding and subtracting algebraic fractions.

Review Exercise

Simplify.

a. $\frac{a}{5} + \frac{2a}{5}$ **b.** $\frac{8}{x} - \frac{3}{x}$ **c.** $\frac{7}{3m} + \frac{3}{4m} - \frac{8}{m}$

d. $\frac{9}{x+2} + \frac{1}{x}$ **e.** $\frac{1}{m+2} - \frac{2}{m+3}$ **f.** $\frac{a+3}{a^2-4} + \frac{1}{a+2}$

g. $\frac{2}{x^2-x-2} - \frac{1}{x^2-1}$ **h.** $\frac{1}{x^2-9x+20} - \frac{1}{x^2-11x+30}$

7.1 Multiplying algebraic fractions

Two algebraic fractions can be multiplied in the same way that two numerical fractions are multiplied. Let us consider the following example.

$\frac{x}{2} \times \frac{x}{3}$

What we mean by performing the multiplication is to express this product as a single fraction.

To perform this multiplication, we multiply the numerators and denominators of the two fractions separately, and obtain a single fraction. That is,

$$\frac{x}{2} \times \frac{x}{3} = \frac{x \times x}{2 \times 3}$$

$$= \frac{x^2}{6}$$

If the terms in the numerator and the denominator can be further simplified, by doing the simplification we can express the answer in the simplest form. These simplifications can be done either before multiplying the fractions or after multiplying the fractions. Let us now consider multiplying two fractions where such simplifications are possible.

Consider $\dfrac{8}{a} \times \dfrac{3}{2b}$.

Here, we can cancel the common factor 2, of the numerator 8 of the first fraction and the denominator $2b$ of the second fraction. We perform this simplification as follows.

$$\frac{8}{a} \times \frac{3}{2b} = \frac{{}^{4}\cancel{8}}{a} \times \frac{3}{{}_{1}\cancel{2}b}$$

Now, by multiplying the expressions in the numerators and denominators of the two fractions separately, we get a single fraction as given below.

$$\frac{8}{a} \times \frac{3}{2b} = \frac{4 \times 3}{a \times b}$$

$$= \underline{\underline{\frac{12}{ab}}} \; .$$

We can also cancel the common factors after multiplying the fractions. Consider the following example.

$$\frac{3}{2a} \times \frac{2b}{3} = \frac{6b}{6a}$$

$$= \frac{b}{a}$$

However, by cancelling the common factors before doing this the multiplication, you can minimise long multiplications and divisions. Therefore, doing this is encouraged.

Observe how the following algebraic expressions are simplified.

Example 1

$$\frac{x}{y} \times \frac{4}{5x}$$

$$= \frac{{}^{1}\cancel{x}}{y} \times \frac{4}{5\cancel{x}_{1}} \quad \text{(Dividing by the common factor } x \text{)}$$

$$= \frac{1 \times 4}{y \times 5}$$

$$= \underline{\underline{\frac{4}{5y}}}$$

When multiplying fractions with algebraic expressions in the numerator and the denominator, first factorise the expressions. This is done to cancel the common factors if there are any. Consider the following example.

Example 2

Simplify $\dfrac{2}{x+3} \times \dfrac{x^2+3x}{5}$

$$\frac{2}{x+3} \times \frac{x^2+3x}{5} = \frac{2}{x+3} \times \frac{x(x+3)}{5} \quad \text{(Factorise } x^2+3x \text{)}$$

$$= \frac{2}{x+3} \times \frac{x(x+3)}{5} \quad \text{(Divide by the common factor } x+3\text{)}$$

$$= \frac{2x}{5}$$

Let us now consider a slightly complex example.

Example 3

Simplify $\dfrac{a^2-9}{5a} \times \dfrac{2a-4}{a^2+a-6}$

$$\frac{a^2-9}{5a} \times \frac{2a-4}{a^2+a-6} = \frac{a^2-3^2}{5a} \times \frac{2\,(a-2)}{(a+3)\,(a-2)}$$

$$= \frac{(a-3)\,(a+3)}{5a} \times \frac{2(a-2)}{(a+3)\,(a-2)}$$

$$= \frac{2\,(a-3)}{5a}$$

because a^2-3^2
$= (a-3)\,(a+3)$
because a^2+a-6
$= (a+3)\,(a-2)$

Exercise 7.1

Multiply the following algebraic fractions.

a. $\dfrac{6}{x} \times \dfrac{2}{3x}$ **b.** $\dfrac{x}{5} \times \dfrac{3}{xy}$

c. $\dfrac{2a}{15} \times \dfrac{5}{9}$ **d.** $\dfrac{4m}{5n} \times \dfrac{3}{2m}$

e. $\dfrac{x+1}{8} \times \dfrac{2x}{x+1}$ **f.** $\dfrac{3a-6}{3a} \times \dfrac{1}{a-2}$

g. $\dfrac{x^2}{2y+5} \times \dfrac{4y+10}{3x}$ **h.** $\dfrac{m^2-4}{m+1} \times \dfrac{m^2+2m+1}{m+2}$

i. $\dfrac{x^2-5x+6}{x^2-1} \times \dfrac{x^2-2x-3}{x^2-9}$ **j.** $\dfrac{a^2-b^2}{a^2-2ab+b^2} \times \dfrac{2a-2b}{a^2+ab}$

7.2 Dividing an algebraic fraction by another algebraic fraction

Recall how you obtained the answer when dividing one fraction by another fraction. You multiplied the first fraction by the reciprocal of the second fraction. Similarly, when dividing an algebraic fraction by another algebraic fraction, we can instead multiply the first by the reciprocal of the second.

Before we study how algebraic fractions are divided, let us consider the reciprocal of an algebraic fraction.

Reciprocal of an algebraic fraction

Recall the facts we have leaned regarding the reciprocal of a number. If the product of two numbers is 1, then each number is the reciprocal or the multiplicative inverse of the other number.

Because $2 \times \frac{1}{2} = 1$, reciprocal of 2 is $\frac{1}{2}$ and reciprocal of $\frac{1}{2}$ is 2.

Because $\frac{1}{3} \times 3 = 1$, reciprocal of $\frac{1}{3}$ is 3 and reciprocal of 3 is $\frac{1}{3}$.

Because $\frac{4}{5} \times \frac{5}{4} = 1$, reciprocal of $\frac{4}{5}$ is $\frac{5}{4}$ and reciprocal of $\frac{5}{4}$ is $\frac{4}{5}$.

The reciprocal of an algebraic fraction is also described similarly. That is,
if the product of two algebraic fractions is 1, then each algebraic fraction is the reciprocal of the other.

Let us multiply the two algebraic fractions $\frac{5}{x}$ and $\frac{x}{5}$.

$$\frac{5}{x} \times \frac{x}{5} = \frac{1}{1} = 1.$$

Therefore, $\frac{5}{x}$ is the reciprocal of $\frac{x}{5}$ and $\frac{x}{5}$ is the reciprocal of $\frac{5}{x}$.

Similarly, because

$$\frac{x+1}{y} \times \frac{y}{x+1} = 1,$$

$\frac{x+1}{y}$ is the reciprocal of $\frac{y}{x+1}$ and $\frac{y}{x+1}$ is the reciprocal of $\frac{x+1}{y}$.

Now it should be clear that, the reciprocal of an algebraic fraction can be obtained by simply interchanging the numerator and denominator, as you have done with numerical fractions.

Observe the following algebraic fractions and their reciprocals.

algebraic fraction	reciprocal
$\frac{m}{4}$	$\frac{4}{m}$
$\frac{a}{a+2}$	$\frac{a+2}{a}$
$\frac{x-3}{x^2+5x+6}$	$\frac{x^2+5x+6}{x-3}$

Let us now consider how to divid an algebraic fraction by another.

Example 1

Simplify $\frac{3}{x} \div \frac{4y}{x}$

$\frac{3}{x} \div \frac{4y}{x} = \frac{3}{x} \times \frac{x}{4y}$ (Instead of dividing by $\frac{4y}{x}$ we multiply by its reciprocal $\frac{x}{4y}$)

$= \frac{3}{\cancel{x}} \times \frac{\cancel{x}}{4y}$ (Dividing by the common factor x)

$= \underline{\underline{\frac{3}{4y}}}$ (Multiplying the numerators and denominators separately)

Let us consider a few more examples.

Example 2

Simplify $\frac{a}{b} \div \frac{ab}{4}$

$\frac{a}{b} \div \frac{ab}{4} = \frac{a}{b} \times \frac{4}{ab}$ (Multiplying by the reciprocal)

$= \frac{\cancel{a}}{b} \times \frac{4}{\cancel{a}b}$ (Cancelling a)

$= \underline{\underline{\frac{4}{b^2}}}$

When there are algebraic expressions in both the numerator and the denominator, we can first factor the expressions, so that the common factors can be easily found and cancelled before simplifying.

Look at the following examples.

Example 3

Simplify $\dfrac{3x}{x^2 + 2x} \div \dfrac{5x}{x^2 - 4}$

$$\frac{3x}{x^2 + 2x} \div \frac{5x}{x^2 - 4} = \frac{3x}{x^2 + 2x} \times \frac{x^2 - 4}{5x} \quad \text{(Multiplying by the reciprocal)}$$

$$= \frac{3x}{x(x+2)} \times \frac{(x-2)(x+2)}{5x} \quad \text{(Factoring the expressions and dividing by the common factors)}$$

$$= \underline{\underline{\frac{3(x-2)}{5x}}}$$

Example 4

Simplify $\dfrac{x^2 + 3x - 10}{x} \div \dfrac{x^2 - 25}{x^2 - 5x}$

$$\frac{x^2 + 3x - 10}{x} \div \frac{x^2 - 25}{x^2 - 5x} = \frac{x^2 + 3x - 10}{x} \times \frac{x^2 - 5x}{x^2 - 25}$$

$$= \frac{(x+5)(x-2)}{x} \times \frac{x(x-5)}{(x-5)(x+5)}$$

$$= \frac{x-2}{1}$$

$$= \underline{\underline{x-2}}$$

Exercise 7.2

Simplify the following algebraic fractions.

a. $\dfrac{5}{x} \times \dfrac{10}{x}$

b. $\dfrac{m}{3n} \div \dfrac{m}{2n^2}$

c. $\dfrac{x+1}{y} \div \dfrac{2(x+1)}{x}$

d. $\dfrac{2a-4}{2a} \div \dfrac{a-2}{3}$

e. $\dfrac{x^2+4x}{3y} \div \dfrac{x^2-16}{12y^2}$

f. $\dfrac{p^2+pq}{p^2-pr} \div \dfrac{p^2-q^2}{p^2-r^2}$

g. $\dfrac{m^2-4}{m+1} \div \dfrac{m+2}{m^2+2m+1}$

h. $\dfrac{x^2y^2+3xy}{4x^2-1} \div \dfrac{xy+3}{2x+1}$

i. $\dfrac{a^2-5a}{a^2-4a-5} \div \dfrac{a^2-a-2}{a^2+2a+1}$

j. $\dfrac{x^2-8x}{x^2-4x-5} \times \dfrac{x^2+2x+1}{x^3-8x^2} \div \dfrac{x^2+2x-3}{x-5}$

8 Areas of Plane Figures between Parallel Lines

By studying this lesson you will be able to

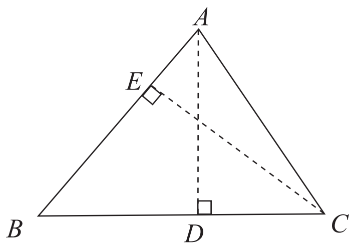
identify the theorems on the relationships between the areas of triangles and parallelograms on the same base and between the same pair of parallel lines, and solve problems related to them.

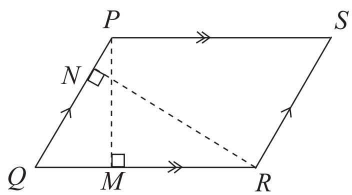
Introduction

You have already learnt about various plane figures and how the areas of certain special plane figures are found. Let us now recall how the areas of triangles and parallelograms are found.

When finding the areas of triangles and parallelograms, the terms altitude and base are used. Let us first recall what these terms mean.

Let us consider the given triangle *ABC* and the parallelogram *PQRS*.





When finding the area of a triangle, any one of its sides can be considered as the base. For example, the side *BC* of the triangle *ABC* can be considered as the base. Then *AD* is the corresponding altitude; that is, the perpendicular dropped from the vertex *A* to the side *BC*.

We know that,

area of triangle $ABC = \frac{1}{2} \times BC \times AD.$

Similarly, if we consider the side *AB* to be the base, the corresponding altitude is *CE*. Accordingly, we can also write,

area of triangle $ABC = \frac{1}{2} \times AB \times CE.$

We can similarly find the area of the triangle *ABC* by taking *AC* as the base and drawing the corresponding altitude from the vertex *B*.

Now let us consider the parallelogram *PQRS*. Here too, the area can be found by considering any one of the sides as the base. If we consider the side *QR* as the base, the corresponding altitude is the line segment *PM*. The length of *PM* is the distance between the two parallel straight line segments *QR* and *PS*, the side opposite *QR*.

We know that,

the area of parallelogram $PQRS = QR \times PM$.

Similarly, if we consider the side *PQ* as the base, the corresponding altitude is *RN*. Therefore we can also write,

the area of parallelogram $PQRS = PQ \times RN$.

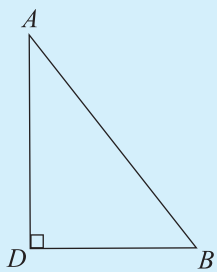
Note
The length of the altitude of a triangle or a parallelogram is also often called the altitude.

To recall what has been learnt earlier regarding finding the areas of parallelograms and triangles, do the following exercise by applying the above facts.

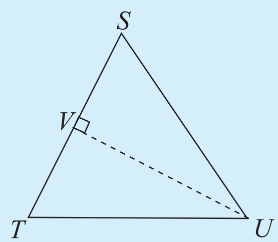
Review Exercise

1. Complete the given table by using the data in each of the figures given below.

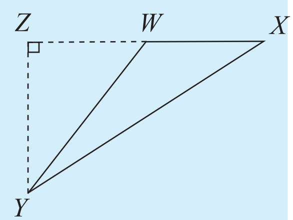
(i)



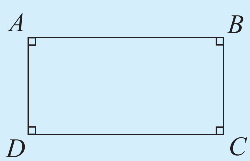
(ii)



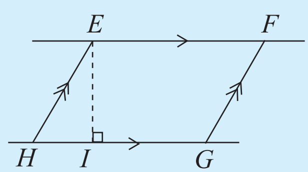
(iii)



(iv)



(v)



(vi)

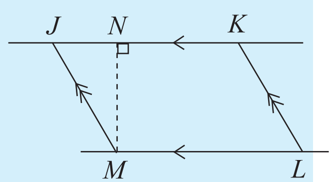


Figure	Base	Corresponding Altitude	Area (As a product of lengths)
(i) Triangle *ABD*			
(ii) Triangle *STU*			
(iii) Triangle *WXY*			
(iv) Rectangle *ABCD*			
(v) Parallelogram *EFGH*			
(v) Parallelogram *JKLM*			

8.1 Parallelograms and triangles on the same base and between the same pair of parallel lines

Let us first see what is meant by parallelograms and triangles on the same base and between the same pair of parallel lines. Consider the following figures.

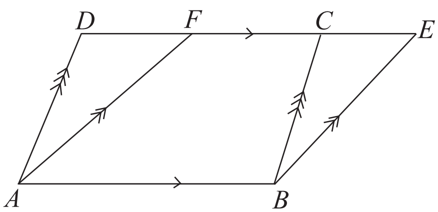


Figure (i)

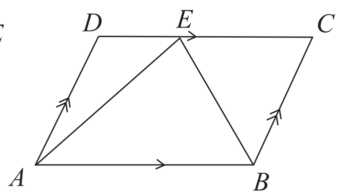


Figure (ii)

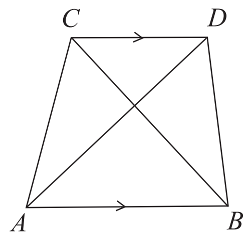


Figure (iii)

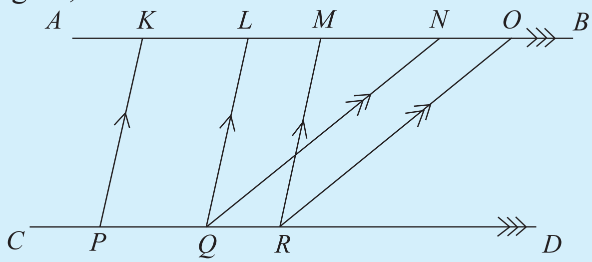
Both the parallelograms *ABCD* and *ABEF* in figure (i) lie between the pair of straight lines *AB* and *DE*. What is meant here by the word “between” is that a pair of opposite sides of each of the parallelograms lies on the straight lines *AB* and *DE*. Further, the side *AB* is common to both parallelograms. In such a situation, we say that the two parallelograms are on the same base and between the same pair of parallel lines. Here, the common side *AB* has been considered as the base. It is clear that corresponding to this common base, both the parallelograms have the same altitude. This is equal to the perpendicular distance between the two parallel lines *AB* and *DE*.

Figure (ii) depicts a parallelogram and a triangle which lie on the same base and between the same pair of parallel lines *AB* and *DC*. The parallelogram is *ABCD* and the triangle is *ABE*. The common base is *AB*. Observe that in this case, one side of the triangle lies on one of the parallel lines while the opposite vertex lies on the other line.

Figure (iii) depicts two triangles on the same base and between the same pair of parallel lines. The two triangles are *ABC* and *ABD*.

Exercise 8.1

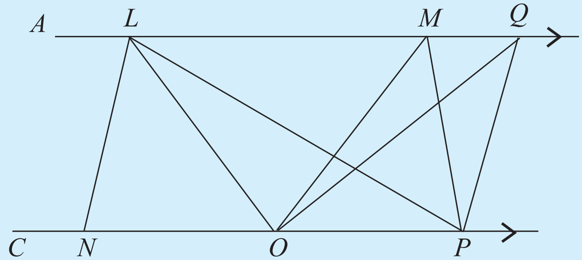
1. Based on the information in the figure,



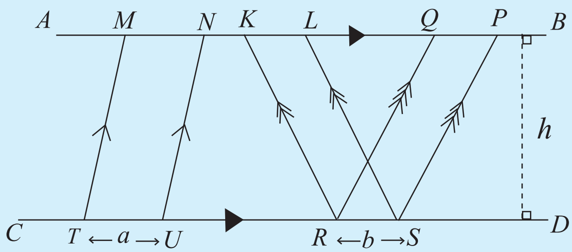
(i) name four parallelograms.

(ii) name the two parallelograms with the same base QR which lie between the pair of parallel lines AB and CD.

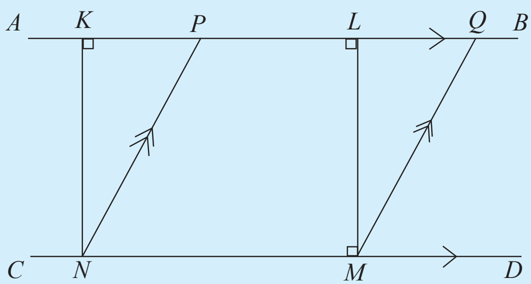
2. Write down all the triangles with the same base OP that lie between the pair of parallel straight lines AQ and CP in the given figure.



3. In the given figure, the perpendicular distance between the pair of parallel straight lines AB and CD is denoted by h and the base lengths of the parallelograms by a and b. Write down the areas of the parallelograms $PQRS$, $KLSR$ and $MNUT$ in terms of these symbols.



4. The rectangle $KLMN$ and the parallelogram $PQMN$ in the given figure lie between the pair of parallel straight lines AB and CD. $NM = 10$ cm and $LM = 8$ cm.



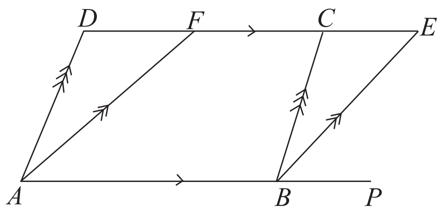
(i) Find the area of the rectangle $KLMN$.

(ii) Find the area of the parallelogram $PQMN$.

(iii) What is the relationship between the area of the rectangle $KLMN$ and the parallelogram $PQMN$?

8.2 The areas of parallelograms on the same base and between the same pair of parallel lines

Next we look at the relationship between the areas of parallelograms on the same base and between the same pair of parallel lines. Consider the given parallelograms.



Let us see whether the areas of the parallelograms $ABCD$ and $ABEF$ are equal.

Observe that,
area of parallelogram $ABCD$ = area of trapezium $ABCF$ + area of triangle AFD
area of parallelogram $ABEF$ = area of trapezium $ABCF$ + area of triangle BEC

Therefore it is clear that, if
the area of triangle AFD = the area of triangle BEC,
then the areas of the two parallelograms will be equal.

In fact, these two triangles are congruent. Therefore their areas are equal. The congruence of the two triangles under the conditions of SAS can be shown as follows.

In the two triangles AFD and BEC,

$AD = BC$ (opposite sides of a parallelogram)
$AF = BE$ (opposite sides of a parallelogram)

Also, since $D\hat{A}B = C\hat{B}P$ (corresponding angles) and $F\hat{A}B = E\hat{B}P$ (corresponding angles), by subtracting these equations we obtain

$D\hat{A}F = C\hat{B}E$.

Accordingly, the two triangles AFD and BEC are congruent under the conditions of SAS.

Therefore we obtain,
area of parallelogram $ABCD$ = area of parallelogram $ABEF$.

We can write this as a theorem as follows.

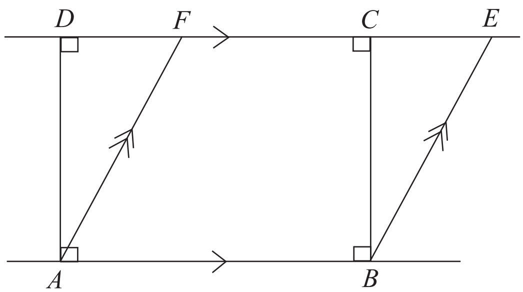
Theorem: Parallelograms on the same base and between the same pair of parallel lines are equal in area.

Now let us obtain an important result using this theorem. You have used the following formula when finding the area of a parallelogram in previous grades and in the above exercise.

Area of a parallelogram = Base $\times$ Perpendicular height

Have you ever thought about how this result was obtained? We can now use the above theorem to prove this result.

The figure depicts a rectangle $ABCD$ (that is, a parallelogram) and a parallelogram $ABEF$ on the same base and between the same pair of parallel lines. According to the above theorem, their areas are equal.

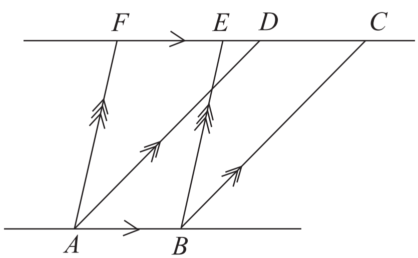


We know that,
area of the parallelogram $ABEF$ = Area of the rectangle $ABCD$
$= AB \times AD$
$= AB \times$ perpendicular distance between the two parallel lines
= base of the parallelogram $\times$ perpendicular height

Let us now consider how calculations are done using this theorem.

Example 1

The area of the parallelogram $ABEF$ in the figure is 80 cm^2 while $AB = 8$ cm.



(i) Name the parallelograms in the figure that lie on the same base and between the same pair of parallel lines.
(ii) What is the area of the parallelogram $ABCD$?
(iii) Find the perpendicular distance between the parallel lines AB and FC.

Now let us answer these questions.

(i) $ABEF$ and $ABCD$.
(ii) Since the parallelograms $ABEF$ and $ABCD$ lie on the same base AB and between the same pair of parallel lines AB and FC, their areas are equal. Therefore, the area of $ABCD = 80 \text{ cm}^2$.
(iii) Let us take the perpendicular distance between the pair of parallel lines as h centimetres.

Then,

area of $ABEF = AB \times h$.

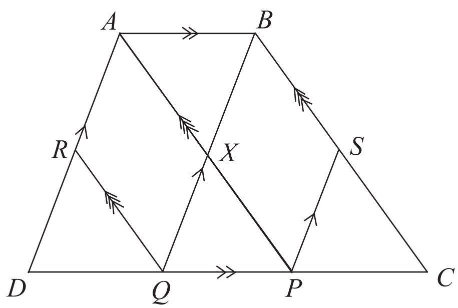
$80 = 8 \times h$

$\therefore \; h = 10$

$\therefore$ the perpendicular distance between the parallel lines is 10 cm.

Now, by considering an example, let us see how riders are proved using this theorem.

Example 2



According to the information in the above figure,

(i) show that $ABQD$ and $ABCP$ are parallelograms.
(ii) show that the parallelograms $ABQD$ and $ABCP$ are of the same area.
(iii) prove that $\Delta\, SPC \equiv \Delta\, RDQ$.
(iv) prove that, area of parallelogram $AXQR$ = area of parallelogram $BXPS$.

(i) In the quadrilateral $ABQD$

$AB // DQ$ (given)

$AD // BQ$ (given)

Since a quadrilateral with pairs of opposite sides parallel, is a parallelogram, $ABQD$ is a parallelogram. Similarly, since $AB // PC$ and $AP // BC$, we obtain that $ABCP$ is a parallelogram.

(ii) Since the parallelograms $ABQD$ and $ABCP$ lie on the same base AB and between the same pair of parallel lines AB and DC, by the above theorem, their areas are equal.

$\therefore$ area of parallelogram $ABQD$ = area of parallelogram $ABCP$.

(iii) In the triangles SPC and RDQ in the figure,

$$S\hat{P}C = R\hat{D}Q \quad \text{(since } SP \,//\, AD\text{, corresponding angles)}$$

$$S\hat{C}P = R\hat{Q}D \quad \text{(since } SC \,//\, RQ\text{, corresponding angles)}$$

Further, $AB = PC$ (opposite sides of the parallelogram $ABCP$)

$AB = DQ$ (opposite sides of the parallelogram $ABQD$)

Therefore, $PC = DQ$.

$\therefore \quad \Delta\, SPC \equiv \Delta\, RDQ$. (AAS)

(iv) Area of parallelogram $ABQD$ = area of parallelogram $ABCP$ (proved)

Area of $\Delta\, RDQ$ = area of $\Delta\, SPC$ (since $\Delta\, RDQ \equiv \Delta\, SPC$)

Therefore,

area of $ABQD$ – area of $\Delta\, RDQ$ = area of $ABCP$ – area of $\Delta\, SPC$.

Then, according to the figure,

area of trapezium $ABQR$ = area of trapezium $ABSP$.

Therefore,

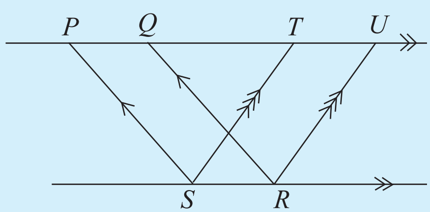
by subtracting the area of the triangle ABX from both sides, we get

area of trapezium $ABQR$ – area of Δ ABX = area of trapezium $ABSP$ – area of Δ ABX

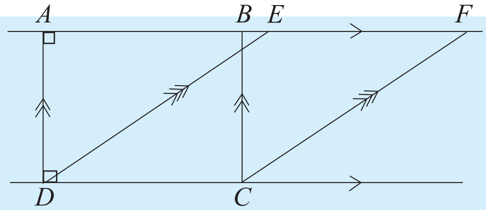
$\therefore$ area of parallelogram $AXQR$ = area of parallelogram $BXPS$.

Exercise 8.2

1. The figure shows two parallelograms that lie between the pair of parallel lines PU and SR. The area of the parallelogram $PQRS$ is 40 cm^2. With reasons, write down the area of the parallelogram $TURS$.



2. A rectangle $ABCD$ and a parallelogram $CDEF$ are given in the figure. If $AD = 7$ cm and $CD = 9$ cm, with reasons, write down the area of the parallelogram $CDEF$.

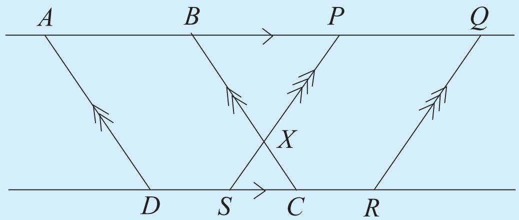


3. The figure shows two parallelograms $ABCD$ and $PQRS$ that lie between the pair of parallel lines AQ and DR. It is given that $DS = CR$.

(i) Show that $DC = SR$.

(ii) Prove that the area of the pentagon $ABXSD$ is equal to the area of the pentagon $PQRCX$.

(iii) Prove that the area of the trapezium $APSD$ is equal to the area of the trapezium $BQRC$.

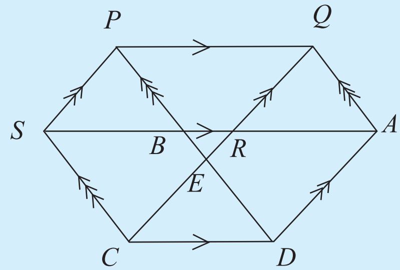


4. Based on the information in the figure,

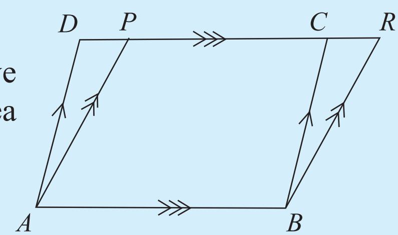
(i) name two parallelograms which are equal in area to the area of the parallelogram $PQRS$.

(ii) name two parallelograms which are equal in area to the area of the parallelogram $ADCR$.

(iii) prove that the area of the parallelogram $PECS$ is equal to the area of the parallelogram $QADE$.



5. Based on the information in the figure, prove that the area of triangle ADP is equal to the area of triangle BRC.



6. Construct the parallelogram $ABCD$ such that $AB = 6$ cm, $D\hat{A}B = 60^0$ and $AD = 5$ cm. Construct the rhombus $ABEF$ equal in area to the area of $ABCD$ and lying on the same side of AB as the parallelogram. State the theorem that you used for your construction.

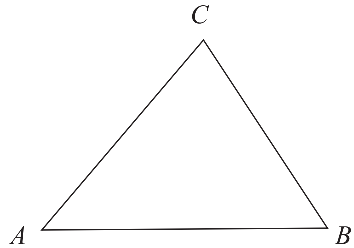
8.3 The areas of parallelograms and triangles on the same base and between the same pair of parallel lines

You have used the following formula in previous grades to find the area of a triangle.

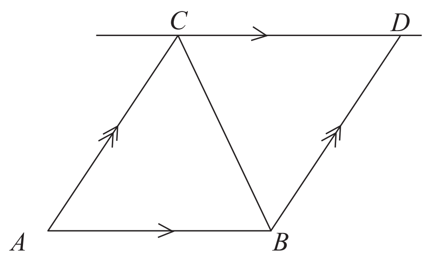
$$\text{Area of the triangle} = \frac{1}{2} \times \text{Base} \times \text{Altitude}$$

Now we will explain why this formula is valid.

Let us consider the following triangle *ABC*.



Now let us draw a line parallel to *AB* through the point *C*, as shown in the figure, and mark the point *D* on this line such that *ABDC* is a parallelogram. In other words let us mark the intersection point of the line drawn through *B* parallel to *AC* and the line drawn through *C* parallel to *AB*, as *D*.



The area of the triangle *ABC* is exactly half the area of the parallelogram *ABDC*. This is because the diagonals of a parallelogram divide the parallelogram into two congruent triangles. We learnt this in the lesson on parallelograms in Grade 10.

Therefore,

$$\begin{aligned} \text{area of triangle } ABC &= \frac{1}{2} \text{ the area of parallelogram } ABDC \\ &= \frac{1}{2} \times AB \times \text{perpendicular distance between } AB \text{ and } CD \\ &= \frac{1}{2} \times AB \times \text{perpendicular height} \end{aligned}$$

We have obtained the familiar formula for the area of a triangle.

Consider again the result that we observed here;

area of triangle $ABC = \frac{1}{2}$ the area of parallelogram $ABDC$.

In section 8.2 of this lesson, we learnt that the areas of parallelograms on the same base and between the same pair of parallel lines are equal. Therefore, in relation to the above figure, the area of any parallelogram that lies on the same base AB and between the same pair of parallel lines AB and CD is equal to the area of $ABDC$. Therefore,

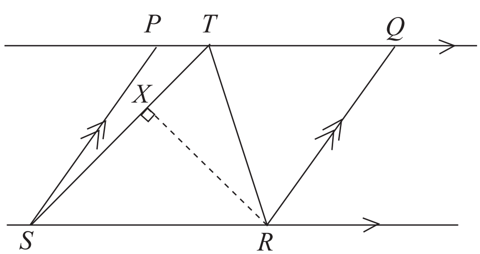
area of triangle $ABC = \frac{1}{2} \times$ (area of any parallelogram with base AB lying between the parallel lines AB and CD).

This result is given below as a theorem.

Theorem: If a triangle and a parallelogram lie on the same base and between the same pair of parallel lines, then the area of the triangle is exactly half the area of the parallelogram.

Let us now consider how calculations are performed using this theorem.

Example 1



The figure illustrates a parallelogram $PQRS$ and a triangle STR on the same base and between the same pair of parallel lines. The area of the parallelogram $PQRS$ is 60 cm^2.

(i) Find the area of the triangle STR. Give reasons for your answer.

(ii) If $ST = 6$ cm, find the length of the perpendicular RX from R to ST.

(i)The parallelogram $PQRS$ and the triangle STR lie on the same base and between the same pair of parallel lines. Therefore the area of triangle STR is half the area of parallelogram $PQRS$.

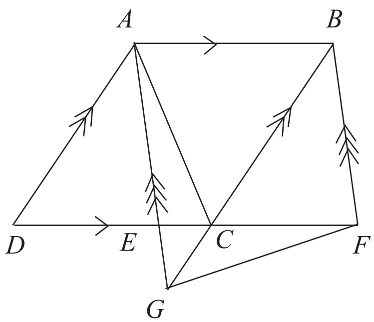
$\therefore$ area of $\Delta\, STR = 30$ cm^2

(ii) Area of $\Delta\, STR = \frac{1}{2} \times ST \times RX$

$$\therefore\ 30 = \frac{1}{2} \times 6 \times RX$$

$$\therefore\ RX = \underline{\underline{10 \text{ cm}}}$$

Example 2



E is a point on the side DC of the parallelogram $ABCD$. The straight line drawn through B parallel to AE, meets DC produced at F. AE produced and BC produced meet at G.

Prove that,

(i) $ABFE$ is a parallelogram.
(ii) the areas of the parallelograms $ABCD$ and $ABFE$ are equal.
(iii) the area of $\Delta\, ACD$ = the area of $\Delta\, BFG$.

(i) In the quadrilateral $ABFE$,
$AE \,//\, BF$ (data)
$AB \,//\, EF$ (data)
$\therefore$ $ABFE$ is a parallelogram (since pairs of opposite sides are parallel)

(ii) The parallelograms $ABCD$ and $ABFE$ lie on the same base AB and between the same pair of parallel lines AB and DF.
$\therefore$ according to the theorem,
area of parallelogram $ABCD$ = area of parallelogram $ABFE$

(iii) The parallelogram $ABCD$ and the triangle ACD lie on the same base DC and between the same pair of parallel lines AB and DC.
$\therefore$ according to the theorem,

$\frac{1}{2}$ the area of parallelogram $ABCD$ = area of triangle ACD.

Similarly,

the parallelogram *ABFE* and the triangle *BFG* lie on the same base *BF* and between the same pair of parallel lines *BF* and *AG*.
Therefore,

$\frac{1}{2}$ the area of parallelogram $ABFE$ = area of triangle BFG

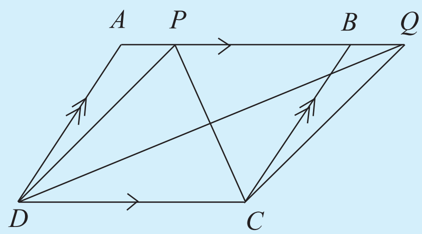
Since, area of parallelogram $ABCD$ = area of parallelogram $ABFE$,

$\frac{1}{2}$ the area of parallelogram $ABCD$ = $\frac{1}{2}$ the area of parallelogram $ABFE$

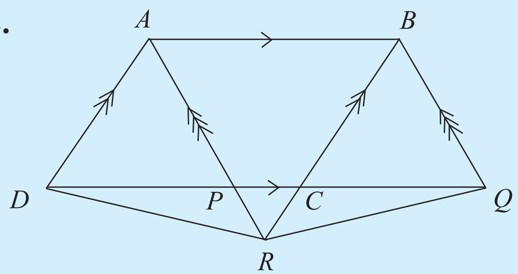
$\therefore$ area of $\Delta\, ACD$ = area of $\Delta\, BFG$

Exercise 8.3

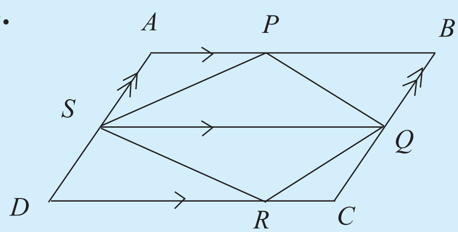
1. The area of the parallelogram *ABCD* in the figure is 50 cm^2.



(i) What is the area of triangle *PDC?*
(ii) What is the area of triangle *DCQ?*

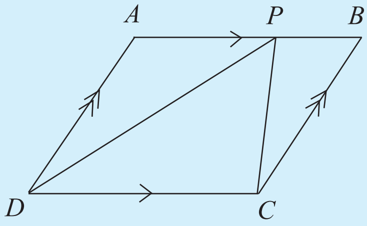
2. 

The point *P* lies on the side *DC* of the parallelogram *ABCD*. The straight line drawn through *B* parallel to *AP* meets *DC* produced at *Q*. Further, *AP* produced and *BC* produced meet at *R*. Prove that the area of triangle *ADR* is equal to the area of triangle *BQR*.

3. 

In the figure, *SQ* has been drawn parallel to the side *AB* of the parallelogram *ABCD*, such that it meets the side *AD* at *S* and the side *BC* at *Q*. Prove that the area of the quadrilateral *PQRS* is exactly half the area of the parallelogram *ABCD*.

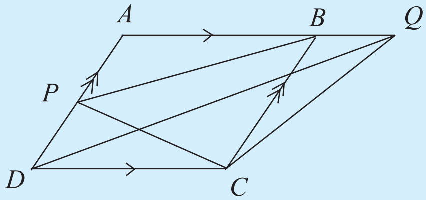
4.



P is any point on the side AB of the parallelogram $ABCD$. Prove that,

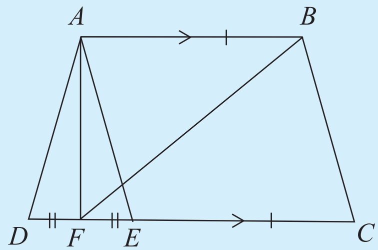
area of $\Delta\, APD$ + area of $\Delta\, BPC$ = area of $\Delta\, DPC$

5.



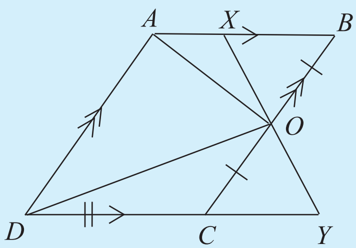
In the figure, the point P lies on the side AD of the parallelogram $ABCD$, and the point Q lies on AB produced. Prove that, area of $\Delta\, CPB$ = area of $\Delta\, CQD$.

6.



$AB//DC$ and $DC > AB$ in the trapezium $ABCD$.
The point E lies on the side CD such that $AB = CE$. The point F lies on the side DE such that the area of the triangle AFE is equal to the area of the triangle ADF. Prove that the area of the trapezium $ABFD$ is exactly half the area of the trapezium $ABCD$.

7.

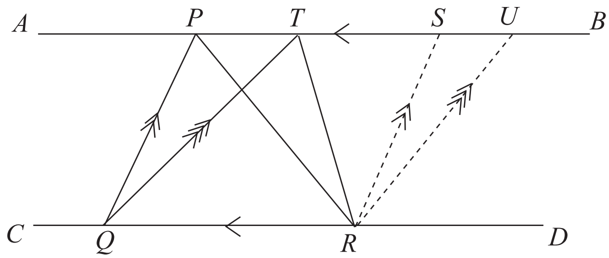


O is the midpoint of the side BC of the parallelogram $ABCD$ and X is an arbitrary point on AB. Also, XO produced and DC produced meet at Y.

Prove that,
(i) the area of $\Delta\, BOX$ = the area of $\Delta\, COY$
(ii) the area of trapezium $AXYD$ = the area of parallelogram $ABCD$.
(iii) the area of trapezium $AXYD$ is twice the area of triangle ADO.

8.4 Triangles on the same base and between the same pair of parallel lines

Now let us consider the two triangles PQR and TQR that lie on the same base QR and between the same pair of parallel lines AB and CD.



As discussed in section 8.3 the parallelogram related to the triangle PQR is $PQRS$, and the parallelogram related to the triangle TQR is $TQRU$.

Since the parallelogram related to the triangle PQR is $PQRS$,

area of triangle $PQR = \frac{1}{2}$ the area of parallelogram $PQRS$.

Since the parallelogram related to the triangle TQR is $TQRU$,

area of triangle $TQR = \frac{1}{2}$ the area of parallelogram $TQRU$.

However, since the parallelograms $PQRS$ and $TQRU$ lie on the same base QR and between the same pair of parallel lines, by the theorem,
area of parallelogram $PQRS$ = area of parallelogram $TQRU$.

$\therefore\ \frac{1}{2}$ the area of parallelogram $PQRS = \frac{1}{2}$ the area of parallelogram $TQRU$

That is, area of triangle PQR = area of triangle TQR.

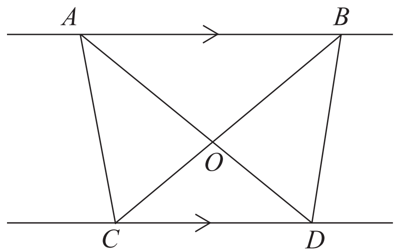
As stated previously, the areas of the two triangles PQR and TQR which lie on the same base QR and between the same pair of parallel lines AB and CD are equal in area.

Triangles which satisfy the above given conditions in this manner are equal in area. This is stated as a theorem as follows.

Theorem: Triangles on the same base and between the same pair of parallel lines are equal in area.

Let us now consider through the following examples how problems are solved using this theorem.

Example 1



In the given figure, $AB//CD$.

(i) Name a triangle that has the same area as triangle ACD. Write down the theorem that your answer is based on.

(ii) If the area of triangle ABC is 30 cm^2, find the area of triangle ABD.

(iii) Prove that the area of triangle AOC is equal to the area of triangle BOD.

(i) Triangle BCD.

Triangles on the same base and between the same pair of parallel lines are equal in area.

(ii) Area of triangle $ABD = 30$ cm^2.

(iii) Area of $\Delta\, ACD$ = Area of $\Delta\, BCD$. (On the same base CD and $AB//CD$.)

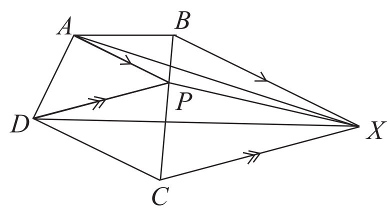
According to the figure, the triangle COD is common to both these triangles. When this portion is removed,

area of $\Delta\, ACD$ – area of $\Delta\, COD$ = area of $\Delta\, BCD$ – area of $\Delta\, COD$

$\therefore$ area of $\Delta\, AOC$ = area of $\Delta\, BOD$

Example 2

The point P lies on the side BC of the quadrilateral $ABCD$. The line drawn though B parallel to AP meets the line drawn through C parallel to DP at X. Prove that the area of triangle ADX is equal to the area of quadrilateral $ABCD$.



Proof: Since the triangles APB and APX lie on the same base AP and between the same pair of parallel lines AP and BX, according to the theorem,

$$\Delta APB = \Delta APX \text{ ———— } (1)$$

Similarly, since $DP//CX$,

$$\Delta DPC = \Delta DPX \text{ ———— } (2)$$

From (1) + (2), $\Delta ABP + \Delta DPC = \Delta APX + \Delta DPX.$

Let us add the area of triangle ADP to both sides.

Then, $\Delta ABP + \Delta DPC + \Delta ADP = \Delta APX + \Delta DPX + \Delta ADP$

$\therefore$ area of quadrilateral $ABCD$ = area of the triangle ADX

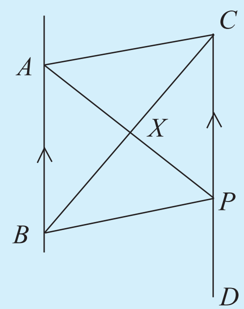
Exercise 8.4

1. The area of triangle ABP which lies between the parallel lines AB and CD in the figure is 25 cm^2.

(i) What is the area of triangle ABC ?

(ii) If the area of triangle ABX is 10 cm^2, what is the area of triangle ACX ?

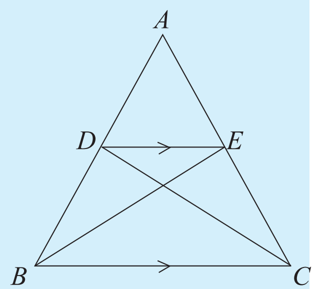
(iii) Explain with reasons what the relationship between the areas of the triangles ACX and BPX is.



2. In the figure, DE is drawn parallel to the side BC of the triangle ABC, such that it touches the side AB at D and the side AC at E.

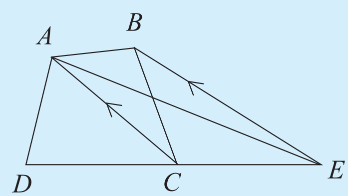
(i) Name a triangle which is equal in area to the triangle BED.

(ii) Prove that the triangles ABE and ADC are equal in area.



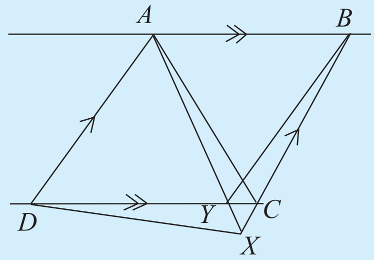
3. The straight line drawn through the point B parallel to the diagonal AC of the quadrilateral $ABCD$, meets the side DC produced at E.

(i) Name a triangle which is equal in area to the triangle ABC. Give reasons for your answer.



(ii) Prove that the area of the quadrilateral $ABCD$ is equal to the area of the triangle ADE.

4. *ABCD* is a parallelogram. A straight line drawn from *A* intersects the side *DC* at *Y* and *BC* produced at *X*.
Prove that,
(i) the triangles *DYX* and *AYC* are equal in area.
(ii) the triangles *BCY* and *DYX* are equal in area.



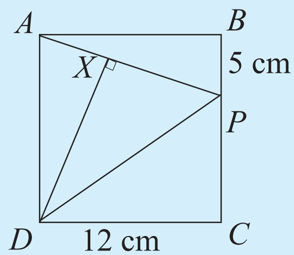
5. The point *Y* lies on the side *BC* of the parallelogram *ABCD*. The side *AB* produced and *DY* produced meet at *X*. Prove that the area of triangle *AYX* is equal to the area of triangle *BCX*.

6. *BC* is a fixed straight line segment of length 8 cm. With the aid of a sketch, describe the locus of the point *A* such that the area of triangle *ABC* is 40 cm^2.

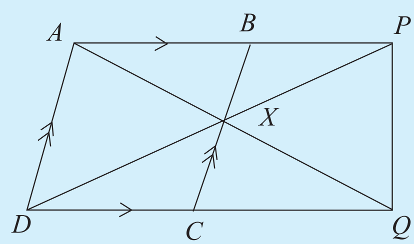
7. Construct the triangle *ABC* such that *AB* = 8 cm, *AC* = 7 cm and *BC* = 4 cm. Construct the triangle *PAB* which is equal in area to the triangle *ABC*, with *P* lying on the same side of *AB* as *C*, and *PA* = *PB*.

Miscellaneous Exercise

1. The length of a side of the square *ABCD* in the figure is 12 cm. The point *P* lies on the side *BC* such that *BP* = 5 cm. Find the length of *DX*.



2. *X* is a point on the side *BC* of the parallelogram *ABCD*. The side *AB* produced and *DX* produced meet at *P* and the side *DC* produced and *AX* produced meet at *Q*. Prove that the area of the triangle *PXQ* is exactly half of the area of the parallelogram *ABCD*.



3. The diagonals of the parallelogram *PQRS* intersect at *O*. The point *A* lies on the side *SR*. Find the ratio of the area of the triangle *POQ* to that of the triangle *PAQ*.

4. *ABCD* and *ABEF* are two parallelograms, unequal in area, drawn on either side of *AB*.
Prove that,
(i) *DCEF* is a parallelogram.
(ii) the area of the parallelogram *DCEF* is equal to the sum of the areas of the parallelograms *ABCD* and *ABEF*.

5. *ABCD* is a parallelogram. *EF* has been drawn parallel to *BD* such that it intersects the side *AB* at *E* and the side *AD* at *F*.
Prove that,
(i) the triangles *BEC* and *DFC* are equal in area.
(ii) the triangles *AEC* and *AFC* are equal in area.

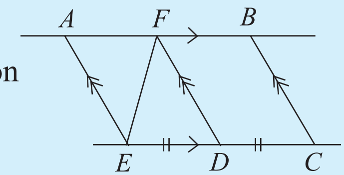
Review Exercise – Term 1

Part 1

1. Simplify $2\sqrt{3} - \sqrt{3}$

2. If $10^{0.5247} = 3.348$ find the value of lg 0.3348.

3. According to the information in the figure, what fraction of the area of *ABCE* is the area of *AFE?*



4. If $A^3 = x^3 - y^3 - 3x^2y + 3xy^2$ express A in terms of x and y.

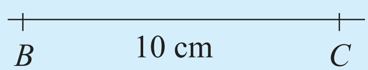
5. A new solid is constructed by pasting together the square bases of two identical square based right pyramids. If the surface area of the new solid is 384 cm², find the area of a triangular face of each pyramid.

6. Simplify: $\dfrac{2}{x-1} - \dfrac{1}{1-x}$

7. Evaluate: $\log_3 27 - \log_4 16$

8. The mass of a sphere made of a special type of material is 120 g. If the mass of 1 cm³ of the material is 4g, find the volume of the sphere.

9. *B* and *C* in the figure are two fixed points that lie 10 cm from each other. Sketch the loci of the point *A* such that the area of the triangle *ABC* is 20 cm².

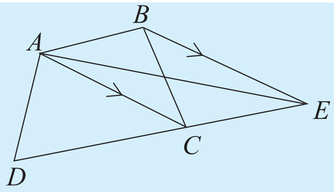


10. If lg 5 = 0.6990 find the value of lg 20.

11. Show that the area of the curved surface of a cylinder of height the length of its diameter, is equal to the surface area of a sphere of the same diameter.

12. Find the value of $\sqrt{20}$ by taking that $\sqrt{5} = 2.23$.

13. Show that the area of the quadrilateral $ABCD$ in the figure is equal to the area of the triangle ADE.



14. Evaluate: $\sqrt{75} \times 2\sqrt{3}$.

15. Simplify: $\dfrac{3x}{x^2-1} \times \dfrac{x(x-1)}{3}$

Part II

1. (i) If $x + \dfrac{1}{x} = 3$ then find the value of $x^3 + \dfrac{1}{x^3}$.

(ii) Simplify: $\dfrac{m^2 - 4n^2}{mn(m+2n)} \div \dfrac{m^2 - 4mn + 4n^2}{m^2n^2}$

2. (i) For what value of x is $2 \lg x = \lg 3 + \lg (2x - 3)$

(ii) If $2 \lg x + \lg 32 - \lg 8 = 2$ determine x.

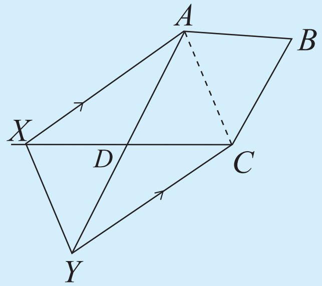
(iii) Find the value without using the logarithms table.

$$\lg_2 \frac{3}{4} - 2 \lg_2 \left(\frac{3}{16}\right) + \lg 12 - 2$$

(iv) Simplify using the logarithms table and give the answer to the nearest second decimal.

$$\frac{\sqrt{0.835} \times 0.75^2}{4.561}$$

3. (a) The side CD of the parallelogram $ABCD$ in the figure has been produced to X. The line drawn through C parallel to AX, meets the side AD produced at Y.



(i) Name a triangle which is equal in area to the triangle AXY. Give reasons for your answer.

(ii) Prove that the area of the triangle XDY is half the area of the parallelogram $ABCD$.

(b) By using only a pair of compasses and a straight edge with a cm/mm scale,

(i) construct the triangle ABC such that $AB = 5.5$ cm, $A\hat{B}C = 60^{\circ}$ and $BC = 4.2$ cm.

(ii) construct the rhombus $ABPQ$ of area twice that of the area of triangle ABC.

4. O is any point on the side BC of the parallelogram $ABCD$. The line drawn through A parallel to DO meets CB produced at P. AO produced meets DC produced at Q.

(i) Based on the above information, sketch a figure and include the given data.

(ii) Write down the relationship between the area of the parallelogram $ABCD$ and the area of the triangle ADO.

(iii) Prove that the area of triangle ABP is equal to the area of triangle BOQ.

5. The base radius and perpendicular height of a solid right circular cone are respectively 7 cm and 12 cm.

(i) Find the volume of the cone.

(ii) If the base radius of the cone is kept fixed and the perpendicular height is doubled, how many times more would the volume of the new cone be than that of the original cone?

(iii) If the perpendicular height is kept fixed and the base radius is doubled, how many times more would the volume of the new cone be than that of the original cone?

ලඝුගණක

மடக்கைகள்

LOGARITHMS

											මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

ලඝුගණක
மடக்கைகள்
LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences 1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Glossary

A

Algebraic Fractions	වීජීය භාග	அட்சரகணிதப் பின்னங்கள்
Area	වර්ගඵලය	பரப்பளவு

B

Bar	වියුති	பிரிகோடு
Base	පාදය	அடி
Binomial Expression	ද්විපද ප්‍රකාශන	ஈருறுப்புக் கோவை

C

Characteristic	පූර්ණාංශය	சிறப்பியல்பு
Circular	වෘත්තාකාර	வட்ட வடிவான
Circumference	පරිධිය	பரிதி
Common denominator	පොදු හරය	பொதுப் பகுதி
Cone	කේතුව	கூம்பு
Cubed	ඝනායිතය	கன
Curved Surface	වක්‍ර පෘෂ්ඨය	வளை மேற்பரப்பளவு

D

Denominator	හරය	Denominator
Division	බෙදීම	வகுத்தல்

E

Entire surds	අඛිල කරණී	முழுமைச் சேடு
Expansion	ප්‍රසාරණය	விரிவு

F

Finite decimals	අන්ත දශම	முடிவுறு தசமம்

I

Indices	දර්ශක	சுட்டி
Infinite decimals	අනන්ත දශම	முடிவில் தசமம்
Integers	නිඛිල	நிறைவெண்கள்
Irrational numbers	අපරිමේය සංඛ්‍යා	விகிதமுறா எண்கள்

L

Least common multiple	කුඩාම පොදු ගුණාකාරය	பொதுமடங்குகளுள் சிறியது
Logarithm	ලඝුගණක	மடக்கை

M

Mantissa	දශමාංශය	தசமக் கூட்டு
Multiplication	ගුණ කිරීම	பெருக்கல்

N

Numerator	ලවය	தொகுதி

P

Parallel lines	සමාන්තර රේඛා	சமாந்தரக் கோடுகள்
Parallelogram	සමාන්තරාස්‍රය	இணைகரம்
Perpendicular height	ලම්බ උස	செங்குத்துயரம்
Power	බලය	வலு
Prism	ප්‍රිස්මය	அரியம்
Pyramid	පිරමීඩය	கூம்பகம்

R

Radius	අරය	ஆரை
Rational numbers	පරිමේය සංඛ්‍යා	
Real numbers	තාත්වික සංඛ්‍යා	மெய் எண்கள்
Reciprocal	පරස්පරය	நிகர்மாறு
Recurring decimals	සමාවර්ත දශම	மீளும் தசமம்
Right circular cone	ඍජු වෘත්ත කේතුව	செவ்வட்டக்கூம்பு
Right pyramid	ඍජු පිරමීඩය	செங்கூம்பகம்

S

Scientific calculator	විද්‍යාත්මක ගණක යන්ත්‍රය	விஞ்ஞானமுறைக் கணிகருவி
Scientific notation	විද්‍යාත්මක අංකනය	விஞ்ஞானமுறைக் குறிப்பீடு
Slant height	ඇළ උස	சாய் உயரம்
Sphere	ගෝලය	கோளம்
Square shape	සමචතුරස්‍රාකාර	சதுர வடிவான

Squared	වර්ගායිතය	வர்க்கம்
Surds	කරණි	சேடு
Surface Area	පෘෂ්ඨ වර්ගඵලය	மேற்பரப்பளவு

T

Term	පදය	உறுப்பு
Theorem	ප්‍රමේයය	தேற்றம்
Triangle	ත්‍රිකෝණය	முக்கோணி
Triangular	ත්‍රිකෝණාකාර	முக்கோண வடிவான
Trignometric Ratio	ත්‍රිකෝණමිතික අනුපාත	திரிகோண விகிதங்கள்

V

Volume	පරිමාව	கனவளவு

MATHEMATICS

Grade 11

Part - II

Educational Publications Department

To obtain electronic textbooks, visit
www.edupub.gov.lk

First Print 2015
Second Print 2016
Third Print 2017
Fourth Print 2018
Fifth Print 2019

All Rights Reserved

ISBN 978-955-25-0310-8

Published by Educational Publications Department
Printed by State Printing Corporation
Panaluwa, Padukka.

The National Anthem of Sri Lanka

Sri Lanka Matha
Apa Sri Lanka Namo Namo Namo Namo Matha
Sundara siri barinee, surendi athi sobamana Lanka
Dhanya dhanaya neka mal palaturu piri jaya bhoomiya ramya
Apa hata sepa siri setha sadana jeewanaye matha
Piliganu mena apa bhakthi pooja Namo Namo Matha
Apa Sri Lanka Namo Namo Namo Namo Matha
Oba we apa vidya
Obamaya apa sathya
Oba we apa shakthi
Apa hada thula bhakthi
Oba apa aloke
Apage anuprane
Oba apa jeevana we
Apa mukthiya oba we
Nava jeevana demine, nithina apa pubudukaran matha
Gnana veerya vadawamina regena yanu mana jaya bhoomi kara
Eka mavakage daru kela bevina
Yamu yamu vee nopama
Prema vada sema bheda durerada
Namo, Namo Matha
Apa Sri Lanka Namo Namo Namo Namo Matha

අපි වෙමු එක මවකගෙ දරුවෝ
එක නිවසෙහි වෙසෙනා
එක පාටැති එක රුධිරය වේ
අප කය තුළ දුවනා

එබැවිනි අපි වෙමු සොයුරු සොයුරියෝ
එක ලෙස එහි වැඩෙනා
ජීවත් වන අප මෙම නිවසේ
සොඳින සිටිය යුතු වේ

සැමට ම මෙත් කරුණා ගුණෙනී
වෙළී සමගි දමිනී
රන් මිණි මුතු නො ව එය ම ය සැපතා
කිසි කල නොම දිරනා

ආනන්ද සමරකෝන්

ஒரு தாய் மக்கள் நாமாவோம்
ஒன்றே நாம் வாழும் இல்லம்
நன்றே உடலில் ஓடும்
ஒன்றே நம் குருதி நிறம்

அதனால் சகோதரர் நாமாவோம்
ஒன்றாய் வாழும் வளரும் நாம்
நன்றாய் இவ் இல்லினிலே
நலமே வாழ்தல் வேண்டுமன்றோ

யாவரும் அன்பு கருணையுடன்
ஒற்றுமை சிறக்க வாழ்ந்திடுதல்
பொன்னும் மணியும் முத்துமல்ல - அதுவே
யான்று மழியாச் செல்வமன்றோ.

ஆனந்த சமரக்கோன்
கவிதையின் பெயர்ப்பு.

Being innovative, changing with right knowledge
Be a light to the country as well as to the world.

Message from the Hon. Minister of Education

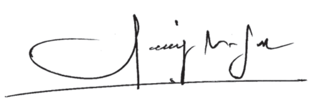
The past two decades have been significant in the world history due to changes that took place in technology. The present students face a lot of new challenges along with the rapid development of Information Technology, communication and other related fields. The manner of career opportunities are liable to change specifically in the near future. In such an environment, with a new technological and intellectual society, thousands of innovative career opportunities would be created. To win those challenges, it is the responsibility of the Sri Lankan Government and myself, as the Minister of Education, to empower you all.

This book is a product of free education. Your aim must be to use this book properly and acquire the necessary knowledge out of it. The government in turn is able to provide free textbooks to you, as a result of the commitment and labour of your parents and elders.

Since we have understood that the education is crucial in deciding the future of a country, the government has taken steps to change curriculum to suit the rapid changes of the technological world. Hence, you have to dedicate yourselves to become productive citizens. I believe that the knowledge this book provides will suffice your aim.

It is your duty to give a proper value to the money spent by the government on your education. Also you should understand that education determines your future. Make sure that you reach the optimum social stratum through education.

I congratulate you to enjoy the benefits of free education and bloom as an honoured citizen who takes the name of Sri Lanka to the world.



Akila Viraj Kariyawasam

Minister of Education

Foreword

The educational objectives of the contemporary world are becoming more complex along with the economic, social, cultural and technological development. The learning and teaching process too is changing in relation to human experiences, technological differences, research and new indices. Therefore, it is required to produce the textbook by including subject related information according to the objectives in the syllabus in order to maintain the teaching process by organizing learning experiences that suit to the learner needs. The textbook is not merely a learning tool for the learner. It is a blessing that contributes to obtain a higher education along with a development of conduct and attitudes, to develop values and to obtain learning experiences.

The government in its realization of the concept of free education has offered you all the textbooks from grades 1-11. I would like to remind you that you should make the maximum use of these textbooks and protect them well. I sincerely hope that this textbook would assist you to obtain the expertise to become a virtuous citizen with a complete personality who would be a valuable asset to the country.

I would like to bestow my sincere thanks on the members of the editorial and writer boards as well as on the staff of the Educational Publications Department who have strived to offer this textbook to you.

W. M. Jayantha Wickramanayaka,
Commissioner General of Educational Publications,
Educational Publications Department,
Isurupaya,
Battaramulla.
10.04.2019

Monitoring and Supervision

W. M. Jayantha Wickramanayaka - Commissoner General of Educational Publications
Educational Publications Department

Direction

W.A. Nirmala Piyaseeli - Commissioner Educational Publications (Development)
Educational Publications Department

Co-ordination

Thanuja Maithrie Witharana - Assistant Commissioner
Educational Publications Department

Chandima de Zoysa - Assistant Commissioner (2019 reprinting)
Educational Publications Department

Board of Editors

Dr. Romaine Jayewardene - Senior Lecturer, University of Colombo

Dr. D.K. Mallawa Arachchi - Senior Lecturer, University of Kelaniya

Dr. Sri Dharan - Senior Lecturer, University of Colombo

Dr. Nalin Ganegoda - Senior Lecturer, University of Sri Jayawardenapura

B.D.C Biyanwila - Director of Education, Ministry of Education

G.P.H. Jagath Kumara - Senior Lecturer National Institute of Education

Thanuja Maithrie Witharana - Assistant Commissioner
Educational Publications Department

Board of Writers

Dr. Rochana Meegaskumbura - Senior Lecturer, University of Peradeniya

Dr. J. K. Rathnayake - Senior Lecturer, University of Colombo

Dr. Jayantha Senadeera - Senior Lecturer, Sri Lanka Open University

H.M.A. Jayasena - In Service Adviser (Retired)

Y.V.R. Witharama - In Service Adviser, Zonal Education Office, Dehiowita

W.M.W.C. Walisinghe - In Service Adviser, Zonal Education Office, Kegalle

Ajith Ranasinghe - In Service Adviser, Zonal Education Office, Homagama

Anura Weerasinghe - In Service Adviser, (Priven), Matara District

W.M.D. Lal Wijekantha - Teacher, St. Thomas College, Mt. Lavinia

N. Vakeeshamoorthi - Director of Education (Retired)

R.S.E. Pushparajan - Assistant Director, Zonal Education Office, Puttalam

V. Murali - Teacher Educator Service Zonal Education Office, Vavuniya

Language Editing

M.A.K. Samurdhi - Teacher, Sirimavo Bandaranaike Vidyalaya, Colombo 07

Proof Reading

Edward R. peiris - Engineer (Retired)

Illustrations ,Cover Page and Technical Assistance

B.T. Chathurani Perera
R.D. Thilini Sewwandi - Computer Assistant, Educational Publications Department

Message of the Board of Compilers

This textbook has been compiled in accordance with the new syllabus to be implemented from 2015.

Textbooks are compiled for students. Therefore, we have made an attempt to compile this textbook in a simple and detailed manner making it possible for you to read and understand it on your own.

We have included descriptions, activities and examples to introduce the subject concepts in an attractive manner and to establish them. Moreover, activities are organized from simple to complex to develop an interest to do them.

We have used the terms related to mathematical concepts in accordance with the glossary of technical terms of mathematics compiled by the Department of Official Languages.

Some subject matter learnt during the earlier grades is necessary to learn the subject content in the grade 11 syllabus. Thus, review exercises are included at the beginning of each chapter to revise previous knowledge. You will be prepared by them for the subject content of grade 11.

In addition, students may use the grade 10 book which you have if you need to recall previuos knoweledge.

You will gain maximum benefit from this textbook by reading the chapters and doing the review exercises of each chapter even before your teacher teaches them in the classroom.

We hope that studying mathematics will be an interesting, joyful and productive experience.

Board of Compilers

Contents

09

Percentages

By studying this lesson, you will be able to:

- compute the loan instalment when the interest is computed on the reducing balance.
- compute the interest rate calculated on the reducing balance given the loan instalment.
- solve problems regarding compound interest.

To review what you have learnt so far on percentages do the following problems.

Review Exercise

1. Compute the following percentages.

a. 12% of 800 rupees. **b.** 8% of 1 Kilometre.

c. 2.5% of 1200 g. **d.** 25% of 2.5 Litres.

2. A shopkeeper buys a wristwatch for Rs 500 and sells it for Rs 600. Calculate the profit as a percentage (of the cost price).

3. A person borrows Rs 8000 at an annual simple interest rate of 6%. How much interest will he pay in a year?

4. A person borrows Rs 5000 at an annual simple interest rate of 10%. How much interest will she pay after two years?

5. Sunimal takes out a loan of Rs 10 000 that charges a monthly simple interest rate of 2%. What is the total amount that he will re-pay if he wants to settle the entire loan in 3 months?

Introduction

Our daily household expenses fall into two main categories as capital expenses and recurrent expenses. Recurrent expenses are the expenses we incur on a regular basis and would include our spending on food, clothes, medicine and electricity bills.

Capital expenses are one-time expenses that are not repeated on a regular basis. For instance our spending on purchasing a land, a house, a vehicle, machinery or furniture is a capital expense. Such purchases are generally of significant value and may often require a loan from the place of work or from a financial institution.

Money borrowed as a loan is not paid back in full right away. Rather, it is paid back over a long period of time as monthly partial payments.Usually a loan is expected to be repaid with interest. The part of the loan and the interest that is to be paid every month is referred to as a loan instalment. However, some manufacturers or distributors may sell their merchandise on an interest free installment plan to promote sales.

Example 1

A furniture manufacturer sells a Rs 30 000 worth wooden wardrobe on an interest free scheme of 12 monthly payments.What is the amount paid as an instalment?

$$\text{Instalment amount} = \text{Rs } \frac{30\,000}{12}$$

$$= \underline{\underline{\text{Rs } 2\,500}}$$

Example 2

A festival advance of Rs 5000 is given to employees of government institutions. This interest free advance has to be paid back in 10 equal monthly instalments. If each installment is deducted from the salary, what is the amount deducted from the salary each month?

$$\text{The amount deducted from the monthly salary each month} = \text{Rs } \frac{5\,000}{10}$$

$$= \underline{\underline{\text{Rs } 500}}$$

9.1 Calculating interest on the reducing balance

There are several ways of charging interest on a loan. Calculating the loan interest on the reducing balance is a common method. Let us explore this.

When you borrow money to repay in monthly instalments, or if you purchase an item with a down payment with the understanding that the rest of the money will be paid in instalments, most of the time you will be expected to pay an interest on the loan. Here, the loan is paid in monthly instalments. The interest calculation is based on the outstanding loan balance. That is the balance money that remains in the borrowers hands as the loan is repaid during the loan term. As the borrower repays instalments, the remaining loan balance reduces over time. Interest is then charged only on the loan amount that the borrower still holds. Therefore this method of calculating the interest is called, computing the interest on the reducing balance.

Once the total interest to be paid is calculated, the monthly installments are determined, such that every installment is of the same value.

Study the following examples to understand the method of calculating the loan interest on the reducing balance and the value of an installment.

Example 1

Mr Wickramasinghe has borrowed Rs 30 000 as a business loan from a bank that charges an annual interest rate of 24%. The loan has to be paid back in six equal monthly instalments and the interest is calculated on the reducing balance. Calculate the value of a monthly instalment.

$$\text{The loan amount} = \text{Rs } 30\,000$$

$$\text{Amount due from the principal loan per month without interest} = \text{Rs } \frac{30\,000}{6}$$

$$= \text{Rs } 5\,000$$

In this method, the outstanding loan balance is reduced by Rs 5000 every month and interest is charged only on the loan balance.

$$\text{Annual interest rate} = 24\%$$

$$\text{Monthly interest rate} = 2\%$$

$$\text{Interest charged for the first month} = \text{Rs } 30\,000 \times \frac{2}{100}$$

$$= \text{Rs } 600$$

$$\text{Interest charged for the second month} = \text{Rs } 25\,000 \times \frac{2}{100}$$

$$= \text{Rs } 500$$

$$\text{Interest charged for the third month} = \text{Rs } 20\,000 \times \frac{2}{100}$$

$$= \text{Rs } 400$$

$$\text{Interest charged for the fourth month} = \text{Rs } 15\,000 \times \frac{2}{100}$$

$$= \text{Rs } 300$$

$$\text{Interest charged for the fifth month} = \text{Rs } 10\,000 \times \frac{2}{100}$$

$$= \text{Rs } 200$$

Interest charged for the sixth month $= \text{Rs } 5\,000 \times \frac{2}{100}$

$= \text{Rs } 100$

The total interest paid $= \text{Rs } 600 + 500 + 400 + 300 + 200 + 100$

$= \text{Rs } 2\,100$

Total amount to be paid after 6 months = loan without interest + interest

The total amount paid $= \text{Rs } 30\,000 + 2\,100$

$= \text{Rs } 32\,100$

$\therefore$ value of a monthly installment $= \text{Rs } 32\,100 \div 6$

$= \underline{\underline{\text{Rs } 5\,350}}$

Calculating the interest as in the above method could be lengthy and time consuming. Therefore we adopt the following easier method to calculate the interest.

Interest charged on amount due per month from the principal loan $= \text{Rs } 5\,000 \times \frac{2}{100}$

$= \text{Rs}100$

Therefore,

the total interest paid $= \text{Rs } 100 \times 6 + 100 \times 5 + 100 \times 4 + 100 \times 3 + 100 \times 2 + 100 \times 1$

$= \text{Rs } 100\,(6 + 5 + 4 + 3 + 2 + 1)$

$= \text{Rs } 100 \times 21$

$= \text{Rs } 2\,100$

where 21 is the total number of portions of the loan that is to be paid within the six months. We will call this the number of month units. We calculate the number of month units as

month units $= 6 + 5 + 4 + 3 + 2 + 1 = 21$

This can be thought of as a sum of an arithmetic progression and can be found using the formula $\frac{n}{2}(a + l)$. Then

month units $= \frac{6}{2}(6 + 1)$

$= 3 \times 7$

$= 21$

That is,

$$\text{month units} = \frac{\text{number of instalments}}{2} \times (\text{number of instalments} + 1)$$

Example 2

A television priced at Rs 25 000 can be purchased by making a down payment of Rs 7 000 and paying the remainder by 12 equal monthly installments. If an annual interest rate of 18% is charged on the loan,where the interest is calculated on the reducing balance, find the value of a monthly installment.

$$
\begin{aligned}
\text{Price of the television} &= \text{Rs } 25\,000 \\
\text{The down payment} &= \text{Rs } 7\,000 \\
\therefore \quad \text{Balance to be paid in installments} &= \text{Rs } 25\,000 - 7\,000 \\
&= \text{Rs } 18\,000 \\
\text{Duration of the loan} &= 12 \text{ month} \\
\therefore \quad \text{Portion of the loan paid in one month} &= \text{Rs } 18\,000 \div 12 \\
&= \text{Rs } 1\,500 \\
\therefore \text{ Interest paid for a month unit} &= \text{Rs } 1\,500 \times \frac{18}{100} \times \frac{1}{12} \\
&= \text{Rs } 22.50 \\
\text{The number of month units the loan is paid over} &= \frac{12}{2}(12+1) \\
&= 6 \times 13 \\
&= 78 \\
\therefore \text{ Total interest paid} &= \text{Rs } 22.50 \times 78 \\
&= \text{Rs } 1\,755 \\
\therefore \text{ Total amount paid} &= \text{Rs } 18\,000 + 1\,755 \\
&= \text{Rs } 19\,755 \\
\therefore \quad \text{The amount of an installment} &= \text{Rs } 19\,755 \div 12 \\
&= \underline{\underline{\text{Rs}1\,646.25}}
\end{aligned}
$$

Example 3

The following advertisement was displayed in a shop:

A washing machine worth Rs 30 000 is available for a down payment of Rs 5000 and 10 equal monthly instalments of Rs 2720.

If the interest on the loan was calculated on the reducing loan balance, calculate the interest rate.

Price of the washing machine = Rs 30 000

The down payment = Rs 5 000

Balance to be paid in installments = Rs 30 000 – 5 000

= Rs 25 000

Portion of the loan paid according to the one month = Rs 25 000 $\div$ 10

= Rs 2 500

The amount paid according to the installment plan = Rs 2 720 $\times$ 10

= Rs 27 200

Total interest paid = Rs 27 200 – 25 000

= Rs 2 200

The number of month units the loan is paid over $= \frac{10}{2}(10 + 1)$

= 55

Interest for a monthly unit = Rs 2 200 $\div$ 55

= Rs 40

The annual interest rate $= \frac{40}{2\,500} \times 100\% \times 12$

$= \underline{\underline{19.2\%}}$

Exercise 9.1

1. Sandamini takes a loan of Rs 50 000 from a bank that charges an annual interest rate of 12%. The loan should be repaid in 10 equal monthly instalments.

(i) Find the amount due from the principal loan amount each month.
(ii) Find the interest charged on the principal loan amount each month.
(iii) For how many month units should she pay the interest?
(iv) Find the total interest she should pay under the reducing loan balance
(v) Find the amount of a monthly installment.

2. A government servant can acquire a loan up to ten times of his monthly salary at an annual interest rate of 4.2%. The loan has to be repaid within 5 years in equal monthly instalments. If Nimal draws a salary of Rs 30 000 per month; determine the following.

(i) How much can Nimal acquire as a loan?
(ii) What is the duration of the loan in months?
(iii) If the interest is charged on the reducing balance, calculate the total interest due.
(iv) What is the total amount due under the reducing loan balance?
(v) Find the amount of a monthly installment.

3. A dining table worth Rs 35 000 can be purchased with a Rs 5000 cash down payment and the rest paid in 15 equal monthly instalments. If the loan is charged an 18% annual interest on the reducing loan balance, calculate the amount of a monthly installment.

4. A motor cycle priced at Rs 150 000 for outright purchase can be bought by making a down payment of Rs 30 000 and paying the rest in 2 years in equal monthly installments. If a 24% annual interest rate is charged on the loan, where the interest is calculated on the reducing loan balance, find the amount of a monthly installment.

5. Mr Kumar has acquired a loan of Rs 12 000 which he intends to repay in 6 equal monthly instalments. The amount paid in a monthly installment is Rs 2 100.

(i) Find the amount due from the principal loan amount each month.
(ii) Find the total interest he should pay.
(iii) Find the total amount paid in insallments.
(iv) What is the number of month units?
(v) Find the interest for a month unit.
(vi) Find the annual interst rate.

6. A refrigerator priced at Rs 36000 for outright purchase can be bought by making a down payment of Rs 6000 and paying the rest in 24 equal monthly installments of Rs 1500. If the interest on the loan is calculated on the reducing loan balance, find the annual interest rate.

7. A sewing machine is available at Rs 23 000 for outright purchase. A person going for an installment plan can purchase it by making a down payment of 5 000 and paying the rest in 10 equal monthly installments of Rs 2 000. If the interest on the loan is calculated on the reducing loan balance, find the annual interest rate.

9.2 Compound Interest

An alternative way of calculating interest for an amount of money borrowed or deposited is the compound interest method. Let us explore how the interest is calculated under this method through an example.

At a bank that pays an annual interest rate of 10%, the account statement provided at the end of 3 years, to a person who has maintained a Rs 25 000 fixed account is as follows.

Date	Description	Deposits (Rs)	Interest (Rs)
2013.01.01	cash deposit	25 000.00	_
2013.12.31	interest	_	2 500.00
2014.01.01	balance	27 500.00	_
2014.12.31	interest	_	2 750.00
2015.01.01	balance	30 250.00	_
2015.12.31	interest	_	3 025.00
2016.01.01	balance	33 275. 00	_

According to the above statement the depositor has earned Rs 2 500 as interest in the year 2013. It is clear that the interest earned is 10% of the principal deposit. The sum of the money deposited in 2013 and the interest earned in the year 2013 that amounts to 27 500 is considered as the total in the account on 2014.01.01. Furthermore, the interest earned in the year 2014 is 2750 and it is 10% of the total Rs 27500. It is apparent that the interest earned at the end of the year is added to the account balance and the interest for the next year is calculate on the new balance.

In this manner, when the interest is calculated every year, not only the principal but the previously earned interest also earns interest. The addition of interest to the principal is called compounding and the interest calculation on the compounded amount is called the **compound interest** method.

The compound interest method can be used when calculating interest on a loan as well as on a deposit.

Example 1

If a person takes a loan of Rs 10 000 at a compound interest rate of 10% per year, calculate the total amount required to repay the entire loan in two years.

$$
\begin{aligned}
\text{Loan amount} &= \text{Rs } 10\,000 \\
\text{Compound interest rate} &= 10\% \\
\text{Interest for the first year} &= \text{Rs } 10\,000 \times \frac{10}{100} \\
&= \text{Rs } 1\,000 \\
\text{Loan amount at the end of the first year} &= \text{Rs } 10\,000 + 1\,000 \\
&= \text{Rs } 11\,000 \\
\text{Interest for the second year} &= \text{Rs } 11\,000 \times \frac{10}{100} \\
&= \text{Rs } 1\,100 \\
\text{Loan amount at the end of the second year} &= \text{Rs } 11\,000 + 1\,100 \\
&= \text{Rs } 12\,100
\end{aligned}
$$

As in the above example, compound interest can be calculated separately for each year and added to the loan amount to find the total loan amount due.

Example 2

Amal invests Rs 50 000 for 3 years in a fixed deposit account which pays 6% annual interest compounded yearly. Nimal invests Rs 50 000 in an account which pays 6% annual simple interest. Calculate the amounts received by Amal and Nimal at the end of three years separately.

Total amount received by Amal at the end of the first year $= \text{Rs } 50\,000 \times \frac{106}{100}$

$= \text{Rs } 53\,000.00$

Total amount received by Amal at the end of the second year $= \text{Rs } 53\,000 \times \frac{106}{100}$

$= \text{Rs } 56\,180.00$

Total amount received by Amal at the end of the third year $= \text{Rs } 56\,180 \times \frac{106}{100}$

$= \underline{\underline{\text{Rs } 59\,550.80}}$

Interest received by Nimal at the end of the third year $= \text{Rs } 50\,000 \times \frac{6}{100} \times 3$

$= \text{Rs } 9\,000.00$

Total amount received by Nimal at the end of three years $= \text{Rs } 9\,000 + 50\,000$

$= \underline{\underline{\text{Rs } 59\,000.00}}$

Total amount received by Amal at the end of three years can also be obtained by

$\text{Rs } 50\,000 \times \frac{106}{100} \times \frac{106}{100} \times \frac{106}{100}$

$= \text{Rs } 59\,550.80$.

Exercise 9.2

1. If a person takes a loan of Rs 5 000 at a compound interest rate of 5% per year, calculate the total amount required to repay the entire loan in two years.

2. If a person deposits Rs 6 000 into an account paying 7% annual interest compounded yearly, how much money will be in the account after 2 years?

3. Radha deposits Rs 8 000 in an account paying 12% annual interest compounded yearly. After one year the bank interest rate drops to 10%. How much money will Radha receive in total as interest at the end of 2 years?

4. Hashan and Caseem are two friends. If Hashan lends Rs 25 000 at a simple annual interest rate of 15% and Caseem lends 25 000 at an interest rate of 14% compounded yearly on the same date, calculate who receives more money after three years.

5. At the beginning of a year a person deposits Rs 40 000 in a bank that pays 12% annual interest compounded every six month (semi annually). Compute how much money in total he receives at the end of the year.

6. A person who has loaned a certain amount at 8% interest compounded yearly, receives Rs 432 as compound interest at the end of two years. Find the amount that he loaned.

Miscellaneous Exercise

1. A television is priced at Rs 45 000. A person who buys it outright for cash, receives a discount of 6% and a person going for an instalment plan can make a down payment of Rs 9 000 and pay the remainder in 12 equal monthly instalments. A 24% annual interest rate is charged on the loan, where the interest is calculated on the reducing balance.

(a) What is the total amount paid if the TV is bought outright for cash?
(b) What is the total amount paid if the TV is bought on an instalment plan?
(c) How profitable is it to buy the TV outright for cash than on an installment plan?

2. A person takes a loan of Rs 100 000 at an interest rate of 4.2% compounded annually and deposits it in a bank that pays 8% interest compounded annually. What is the profit of his investment after 2 years?

3. A person takes a loan at a certain interest rate compounded annually. If he has to pay Rs 14 400 to settle the entire loan in 2 years or Rs 17 280 to settle it in 3 years, calculate the amount he borrowed and the compound interest rate.

10 Share Market

By studying this lesson you will be able to

- identify the stock market and its nature
- identify terms related to the stock market
- calculate the dividends gained by investing in the stock market
- solve problems related to shares

Introduction

In this lesson we will consider businesses in Sri Lanka which are incorporated companies registered under the Companies Act No. 07 of 2007. These companies may be owned by an individual or a group of individuals. Of these companies, the companies which are limited by shares can be classified as follows.

- Private limited companies
- Public limited companies

Public limited companies raise capital to commence or continue their business by issuing shares or debentures to the general public. The public is notified regarding the issue of shares through the media. When the public buy shares, they have the right to sell these shares to others. The stock market is where the trading of these shares occurs.

Stock Market

The Stock Market (Stock Exchange) is a place where securities such as shares and debentures issued by companies are traded. The Colombo Stock Exchange (CSE) is the organization responsible for the operation of the stock market in Sri Lanka. The CSE is licensed by the Securities and Exchange Commission of Sri Lanka (SEC) to operate as a stock market in Sri Lanka. The SEC regulates and oversees the stock market. Companies are admitted to the official list of the CSE as listed companies. On April 21st 2015 there were 297 companies listed in the Sri Lankan stock market. The public who wish to buy or sell securities in the stock market must register with one or more of the 15 licensed stockbroker firms of the CSE who attend to the transactions. Some brokers offer trading through the internet.

10.1 Shares

Public limited companies which are listed in the stock market, that wish to raise capital by involving the public, do this by issuing "shares". A share is one of the equal parts into which a company's stated capital is divided.

When a company issues new shares, the price of a share is decided by the company itself. The public can invest in as many shares as they like. An investor who buys shares in a company becomes a part owner of that company. The stake he has in the company is proportional to the number of shares he owns.

To understand this further, consider the following example.

An investor buys 10 000 shares from the 100 000 shares issued to the public by a certain company. Then the investor has a share of $\frac{10\,000}{100\,000}$ in the company. Let us express this as a percentage.

$$\frac{10\,000}{100\,000} \times 100\% \quad = 10\%.$$

Therefore, the investor has 10% ownership of the company.

Example 1

Company C which has a stated capital of Rs 10 000 000, issues 100 000 shares to the public at Rs 100 per share. Vishwa buys 5000 of these shares.

(i) Express Vishwa's share in the company,
   (a) as a fraction
   (b) as a percentage.

(ii) Find the amount that Vishwa invested in company C.

(i) Total number of shares issued by the company = 100 000
Number of shares Vishwa bought = 5 000

(a) Vishwa's share in the company as a fraction $= \frac{5\,000}{100\,000} = \frac{1}{20}$

(*b*) Vishwa's share in the company as a percentage $= \frac{1}{20} \times 100\%$

$= \underline{\underline{5\%}}$

(ii) Price of a share $=$ Rs 100

The number of shares Vishwa purchased $= 5\ 000$

The amount Vishwa invested $=$ Rs $100 \times 5\ 000$

$=$ $\underline{\underline{\text{Rs } 500\ 000}}$

Dividends

When a listed company issues new shares, it gives notice of the benefits that shareholders will enjoy. Dividends are payments out of the earnings of the company which some companies give their shareholders as a benefit. It is expressed in terms of the amount paid per share. This is paid out quarterly or annually.

For example, a company may pay its shareholders annual dividends of Rs 5 per share. The company has the right to change this amount with time.

Let us consider again the above example to clarify this further.

Example 1

Company C pays annual dividends of Rs 4 per share, for the Rs 100 shares it issued, of which Vishwa bought 5 000 shares.

(i) Find the annual income that Vishwa receives through this investment.

(ii) Express Vishwa's annual income as a percentage of the amount he invested.

(i) The number of shares Vishwa owns $= 5000$

Annual dividends per share $=$ Rs 4

$\therefore$ Vishwa's annual income $=$ Rs 5000×4

$=$ $\underline{\underline{\text{Rs } 20\ 000}}$

(ii) Amount that Vishwa invested $=$ Rs $100 \times 5\ 000$

$=$ Rs 500 000

$\therefore$ Vishwa's annual income as a percentage of his investment $= \frac{20\ 000}{500\ 000} \times 100\%$

$= \underline{\underline{4\ \%}}$

Now do the following exercise pertaining to the facts on initial investments in shares.

Exercise 10.1

1. An investor purchases 1000 shares of value Rs 25 per share in the company Sasiri Apparels.

(i) How much did the person invest?

(ii) If the company pays annual dividends of Rs 4 per share, find the investor's annual dividends income.

2. Complete the following tables.

(i)

Price of a share (Rs)	Number of shares	Amount invested (Rs)
10	2500	....
20	5000	....
....	500	50 000
....	4000	80 000
30	....	30 000
45	....	135 000

(ii)

Number of shares	Annual dividends per share (Rs)	Annual dividends income (Rs)
500	2	....
1000	3.50	....
....	5	5000
....	2.50	500 000
2000	....	8000
750	....	2250

3. A public limited company issues 10 000 000 shares to the public at Rs 25 per share to raise its capital. The company pays annual dividends of Rs 5 per share. Sujeeva purchases 50 000 shares in this company.

(i) Find the stated capital of this company.

(ii) Find the amount Sujeeva invests in this company.

(iii) Find the dividends Sujeeva receives annually through this investment in shares.

(iv) What percent of the amount he invested is the annual dividends he receives?

4. Mehela bought a certain number of shares at Rs 20 per share in a company which pays annual dividends of Rs 3 per share. His dividends income at the end of a year from this investment was Rs 12 000.
  (i) Find the number of shares Mahela owns in this company.
  (ii) Find the amount Mahela invested to buy shares in this company.

5. Ganesh spends exactly half of Rs 100 000 to buy a certain number of Rs 25 shares in a company that pays annual dividends of Rs 4 per share. He decides to deposit the remaining amount in a financial institute which pays an annual interest rate of 12%. Show with reasons, which of the two investments is more advantageous.

10.2 Trading in the stock market

We learnt earlier that only listed companies are able to trade their shares in the stock market. Let us consider the following note to learn about the trading of shares that is done after a listed company has issued new shares to the public.

Nethmi Limited Company which pays annual dividends of Rs 2 per share, issued 100 000 shares to the public at Rs 10 per share at its initial offering. In a year, the value of these shares had increased in the stock market to Rs 20 per share, at which time Nadeesha bought 1000 shares. A few years later, when the price of a share in this company had increased to Rs 28, Nadeesha sold her 1000 shares.

The **primary market** is where investors buy new shares issued by a company. Shares can only be bought in the primary market, and the purchases are done directly from the issuing company at the initial price stated by the company itself. However, subsequent to the original issuance of shares in the primary market, trading of shares can be done by investors in the **secondary market**. The price of a share in the secondary market is called the **market price**. The market price of a share varies with time depending on the demand.

In the above example, the price of a share in Nethmi Limited Company increased from Rs 10 to Rs 20 in a year and subsequently after several years to Rs 28. Such increases and decreases in the price of a share occur in the secondary market, where investors are able to trade their shares.

Capital Gain

The price at which a share is bought in a company at the initial offering or later at the market price is called the **purchase price** of the share and the market price at which it is sold is called the **selling price** of the share.

When an investor sells shares he owns he may make a profit or incur a loss.

When he sells his shares,

if the selling price > purchase price, then he makes a capital gain and
the capital gain = selling price – purchase price.

Similarly,

if the selling price < purchase price, then the investor incurs a capital loss and
the capital loss = purchase price – selling price.

Example 1

Mr. Perera who invests in the stock market, bought 2000 shares in a certain company when the market price of a share was Rs 20. When the market price of a share increased to Rs 25, he sold all his shares.

(i) Find the amount Mr. Perera invested in the company.
(ii) Find the amount he made by selling the shares.
(iii) Find his capital gain.
(iv) Express his capital gain as a percentage of his investment.

(i) Amount invested in the company = Rs $20 \times 2\,000$
= $\underline{\underline{\text{Rs } 40\,000}}$

(ii) Amount received by selling the shares = Rs $25 \times 2\,000$
= $\underline{\underline{\text{Rs } 50\,000}}$

(iii) Capital gain = Rs $50\,000 - 40\,000$
= $\underline{\underline{\text{Rs} 10\,000}}$

(iv) The capital gain as a percentage of the amount invested $= \dfrac{10\,000}{40\,000} \times 100\%$

$= \underline{\underline{25\%}}$

The capital gain percentage mentioned in (iv) above can be calculated using the price of a share too.

Purchase price of a share = Rs 20
Selling price of a share = Rs 25

$\therefore$ capital gain as a percentage of the amount invested $= \dfrac{25-20}{20} \times 100\%$

$$= \frac{5}{20} \times 100\%$$

$$= \underline{\underline{25\%}}$$

Example 2

Mr. Mohamed spent a certain amount from the Rs 96 000 he had in hand, to buy shares at Rs 18 per share, in Company A which pays annual dividends of Rs 2 per share. He spent the remaining amount to buy shares at Rs 21 per share, in Company B which pays annual dividends of Rs 3.50 per share. At the end of a year he received Rs 1000 more as annual dividends income from Company B, than the amount he received from Company A.

(i) By taking the amount that Mr. Mohamed invested in Company A as x, construct an equation in x.
(ii) Find the amount he invested in each company by solving the above equation.
(iii) Find the number of shares he had in each company.
(iv) Find the annual dividends income he received from each company.

After receiving the annual income, Mr. Mohamed sold all the shares he owns in both companies at the market price of Rs 20 per share.

(v) Find the total amount he made by selling all his shares in both companies.
(vi) Show that Mr. Mohamed's expectation of making a profit of 20% on his original investments through the dividends income and capital gains was not fulfilled.

(i) Number of shares bought in company A $= \dfrac{x}{18}$

The annual dividends income from company A $=$ Rs $\dfrac{x}{18} \times 2 = \dfrac{x}{9}$

Similarly,

the annual dividends income from company B $=$ Rs $\dfrac{(96\,000 - x)}{21} \times 3.50$

$$= \text{Rs } \frac{(96\,000 - x)}{21} \times \frac{7}{2}$$

$$= \text{Rs } \frac{(96\,000 - x)}{6}$$

$$\therefore \quad \frac{(96\,000 - x)}{6} - \frac{x}{9} = 1000 \text{ is the required equation.}$$

(ii) $$\frac{(96\,000 - x)}{6} - \frac{x}{9} = 1000$$

$$18 \times \frac{(96\,000 - x)}{6} - 18 \times \frac{x}{9} = 18 \times 1000$$

$$3\,(96\,000 - x\,) - 2x = 18\,000$$

$$288\,000 - 3x - 2x = 18\,000$$

$$288\,000 - 18\,000 = 5x$$

$$270\,000 = 5x$$

$$x = 54\,000$$

$\therefore$ amount invested in company A is Rs 54 000.

Amount invested in company B = Rs 96 000 – 54 000 = Rs 42 000

(iii) Number of shares owned in company A $= \dfrac{54\,000}{18} = 3000$

Number of shares owned in company B $= \dfrac{42\,000}{21} = 2000$

(iv) Income received from investment in company A = Rs 3000×2 = Rs 6000

Income received from investment in company B = Rs 2000×3.50 = Rs 7000

(v) Amount received by selling shares in company A = Rs 3000×20 = 60 000

Amount received by selling shares in company B = Rs 2000×20 = 40 000

$\therefore$ total amount received by selling the shares in both the companies = Rs 60 000 + 40 000

= Rs 100 000

The annual dividends income received from both companies = Rs 6000 + 7000

= Rs 13 000

$\therefore$ the sum of the amounts received as dividends income and by selling the shares = Rs 100 000 + 13 000

= Rs 113 000

Amount invested to buy shares in the two companies $= \text{Rs } 96\,000$

Profit $= \text{Rs } 113\,000 - 96\,000$
$= \text{Rs } 17\,000$

$\therefore$ the profit from the investment as a percentage of the amount invested $= \dfrac{17\,000}{96\,000} \times 100\%$

$= 17.7\%$

Since 17.7% < 20%, Mr. Mohamed's expectations were not fulfilled.

Exercise 10.2

1. Complete the following table.

Amount invested (Rs)	Market price of a share (Rs)	Number of shares	Income from annual dividends of Rs 3 per share (Rs)
50 000	25	....	....
....	40	....	1500
75 000	....	....	3000
....	15	500	....
120 000	....	2000	....

2. Tharindu invested Rs 60 000 and bought shares in a company at the market price of Rs 30 per share. The company pays annual dividends of Rs 4 per share.

(i) Find the number of shares Tharindu bought.
(ii) Find the annual dividends income that Tharindu receives from this investment.
(iii) What percentage of the amount invested is the annual dividends income?

3. Ramesh bought 5000 shares in a certain company when the market price was Rs 40 per share. He then sold all these shares when the market price had increased to Rs 50 per share.

(i) Find Ramesh's capital gain per share.
(ii) Find his capital gain due to selling all the shares.
(iii) Express his capital gain as a percentage of the amount invested.

4. A businessman invested Rs 40 000 and bought shares in a certain company at the market price of Rs 40 per share. At the end of a year, he received dividends of 10% on his investment. After receiving this income he sold all his shares at Rs 50 per share.

(i) Find the annual income the businessman received from the company.
(ii) Find the annual dividends the company paid per share.
(iii) Find the amount the businessman received by selling his shares.
(iv) Find his capital gain.
(v) Express his capital gain as a percentage of the amount invested.

5. A person who invested in a company and bought shares at the market price of Rs 20 per share, sold all his shares on an occasion when the market price increased. His capital gain from this sale was 80% of his investment.

(i) What was his capital gain per share?
(ii) At what price did he sell each share?

6. A person bought shares in a company at the market price of Rs 24 per share and sold the shares when the market price per share was Rs 30. Express his capital gain as a percentage of the amount invested.

7. A person bought 1000 shares in a company which pays annual dividends of Rs 6 per share, at the market price of Rs 40 per share. After receiving dividends for a year, he sold his shares on an occasion when the market price of the shares had increased. His total income from the dividends and the sale of the shares was Rs 71 000.

(i) How much was the annual dividends income from this investment?
(ii) What was the selling price of a share?
(iii) Find his capital gain.

8. Devinda invested equal amounts in two companies. He bought shares in one company which pays annual dividends of Rs 4 per share, at the market price of Rs 20 per share, and shares in the second company which pays annual dividends of Rs 5 per share, at the market price of Rs 25 per share. Express his income from each company as a percentage of the amount invested. (Hint: Take the amount that he invested in each company to be Rs x)

9. An investor invested a certain amount from the Rs 70 000 in hand to buy shares in a company which pays annual dividends of Rs 3 per share, at the market price of Rs 30 per share. The rest of the money he invested in a company that pays annual dividends of Rs 4 per share, and bought shares at the market price of Rs 20 per share. If his dividends income for a year from these investments was Rs 9 500, find the amount he invested in each company.

10. An investor, who owned 4000 shares in a company which pays annual dividends of Rs 5 per share, sold all his shares when the market price was Rs 45 per share. He spent all the money he received by selling these shares, to buy shares in a company at the market price of Rs 25 per share. From this investment he gained an annual dividends income which was Rs 8 800 more than what he received from his previous investment. Find the annual dividends per share that the second company paid.

Miscellaneous Exercise

1. Malki placed Rs 50 000 for a year in a fixed deposit, with a financial institute which pays an annual interest rate of 12%. At the end of the year she withdrew the money and used the principal amount and the interest she received to buy shares in a company that pays annual dividends of Rs 4 per share. She bought the shares at the market price of Rs 28 per share.

(i) Find the annual interest Malki received from her fixed deposit.
(ii) Find the amount she invested in shares.
(iii) Find the annual dividends income she received from her investment.
(iv) With reasons state whether it would have been more profitable for Malki to have re-invested the principal amount together with the interest in a fixed deposit with the financial institute for another year, than to have invested in shares.

2. An investor, who owned 1500 shares in a company which pays annual dividends of Rs 2 per share, sold these shares at the market price of Rs 32 per share after receiving the annual dividends income. He invested the money he received by selling the shares, in another company which pays annual dividends of Rs 2 per share. The market price at which he bought the shares in the second company was Rs 40 per share. Show that the ratio of the dividends income from the first company to the dividends income from the second company is 5:4.

3. Udesh took a loan of Rs 40 000 from a financial institute, at an annual simple interest rate of 12%. With this loan, he bought shares at Rs 20 per share, in a company which pays annual dividends of Rs 4.50 per share. After three years, he sold all the shares at the current market price of Rs 28 per share and paid off the loan together with the interest. Show that Udesh made a profit of Rs 28 600 from his investment.

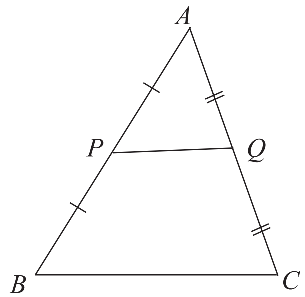
4. Upul invests in a company by buying shares when the market price of a share is Rs 48. He plans to sell the shares when the market price has increased sufficiently, so that his capital gain upon selling the shares will be 30% of his investment. At what price should he sell a share, for him to achieve this?

11 The Midpoint Theorem

By studying this lesson you will be able to
- understand the midpoint theorem and its converse,
- perform calculations and prove riders using the midpoint theorem and its converse.

11.1 The Midpoint Theorem

The midpoint theorem is a result related to the lengths of the sides of a triangle. Let P be the midpoint of the side AB, and Q be the midpoint of the side AC of the triangle ABC.



Then,

$AP = PB$ and $AQ = QC$.

This can also be written as $AP = PB = \frac{1}{2}AB$ and $AQ = QC = \frac{1}{2}AC$.

PQ is the line segment obtained by joining the midpoints of the sides AB and AC.

Theorem
The straight line segment through the midpoints of two sides of a triangle is parallel to the third side and equal in length to half of it.

In relation to the above figure, according to the theorem,

$PQ // BC$ and

$PQ = \frac{1}{2}BC$.

Let us do the following activity to establish this theorem.

Activity 1

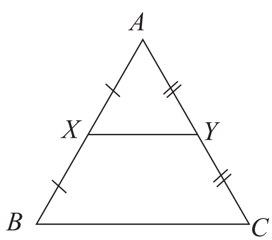
Construct the triangle ABC such that $AB = 6$ cm, $BC = 7$ cm and $CA = 8$ cm. Name the midpoints of AB and AC as P and Q respectively. Join PQ

(i) Measure the length of PQ and establish the fact that it is half the length of BC.
(ii) Check using a set square or by some other method that PQ is parallel to BC.

By doing the above activity, you would have seen that $PQ = \frac{1}{2}BC$ and $PQ//BC$.

Let us consider through an example how the lengths of the sides of rectilinear plane figures related to triangles are found using the midpoint theorem.

Example 1



An equilateral triangle ABC of side length 12 cm is represented in the above figure.The midpoints of AB and AC are X and Y respectively.

Determine the following.

(i) The length of XY.
(ii) The perimeter of the quadrilateral $BCYX$.

(i) According to the midpoint theorem,

$XY//BC$ and $XY = \frac{1}{2}BC$.

$$\therefore\ XY = \frac{1}{2} \times 12$$
$$= 6$$

$\therefore$ the length of XY is 6 cm.

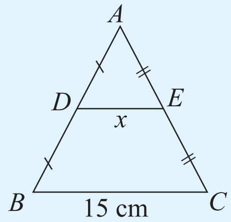
(ii) The perimeter of $BCYX = BC + CY + XY + XB$

$$= 12 + 6 + 6 + 6$$
$$= 30$$

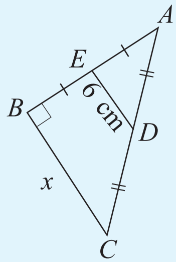
$\therefore$ the perimeter of $BCYX$ is 30 cm.

Exercise 11.1

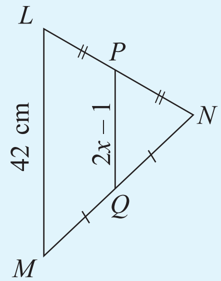
1. Determine the value of x in each figure.



(i)

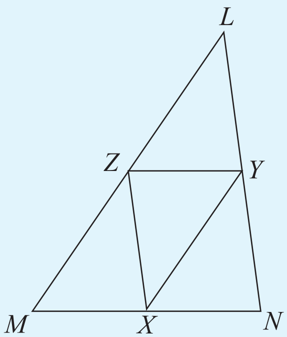


(ii)



(iii)

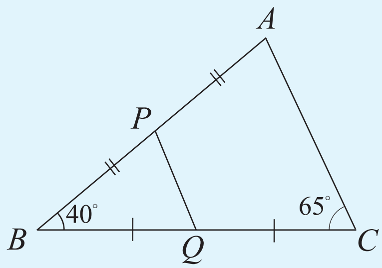
2.



In the given figure, X, Y and Z are the midpoints of the sides MN, NL and LM respectively of the triangle LMN. If $MN = 8$ cm, $NL = 10$ cm and $LM = 12$ cm, find the perimeter of the triangle XYZ.

3. In the quadrilateral $ABCD$, $AC = 15$ cm and $BD = 10$ cm. Find the perimeter of the quadrilateral that is obtained by joining the midpoints of the sides AB, BC, CD and DA.

4.

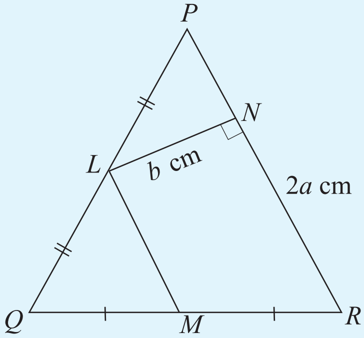


Based on the information in the figure,

(i) if the perimeter of ABC is 22 cm, $AB = 8$ cm and $BC = 10$ cm, find the perimeter of the triangle PBQ.

(ii) if $\hat{B} = 40^\circ$, and $\hat{C} = 65^\circ$, find the remaining angles in the quadrilateral $PQCA$.

5.



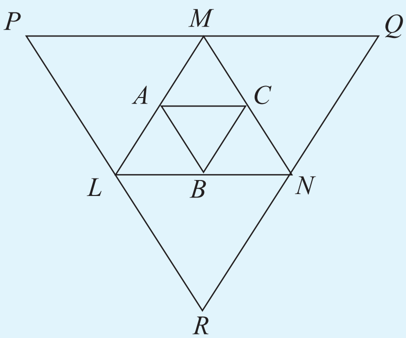
The midpoints of the sides QR and QP of the triangle PQR in the figure are M and L respectively.

$QR + QP = 16$ cm, $PR = 2a$ cm, $LN = b$ cm and $L\hat{N}R = 90^{\circ}$.

(i) Find the perimeter of the quadrilateral $LMRP$ in term of a.

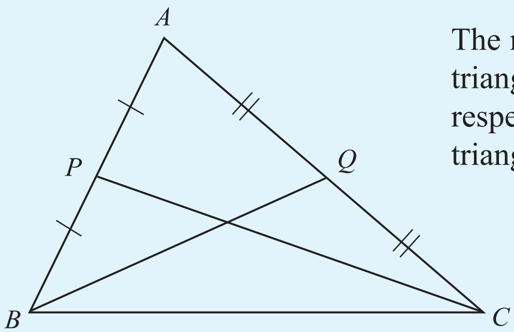
(ii) Express the area of $LMRP$ in terms of a and b.

6.



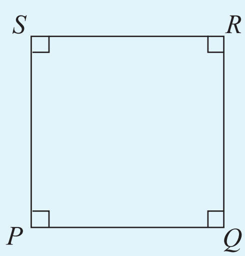
The triangle LMN has been formed by joining the midpoints L, M and N of the sides PR, PQ and QR respectively of the triangle PQR. The triangle ABC has been formed by joining the midpoints A, B and C of the sides LM, LN and MN of the triangle LMN. If the perimeter of the triangle PQR is 12 cm, find the perimeter of the triangle ABC.

7.



The midpoints of the sides AB and AC of the triangle ABC in the figure are P and Q respectively. Show that the areas of the triangles PBC and BQC are equal.

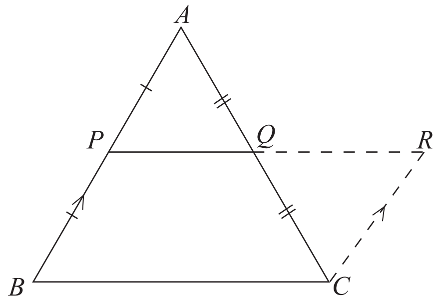
8.



The perimeter of the square $PQRS$ in the figure is 60 cm. Find the perimeter of the quadrilateral that is formed by joining the midpoints of the sides of the square and express it in surd form.

11.2 Proof of the midpoint theorem

Now let us consider the formal proof of the midpoint theorem.



Data: The midpoints of the sides AB and AC of the triangle ABC are P and Q respectively.

To be proved:

$PQ//BC$ and

$PQ = \frac{1}{2} BC.$

Construction: Draw a straight line through C parallel to BP such that it meets PQ produced at R.

Proof:

In the two triangles APQ and $QCR,$

$AQ = QC$ (since Q is the midpoint of AC)

$A\hat{P}Q = Q\hat{R}C$ (since $AP//RC$, alternate angles)

$A\hat{Q}P = R\hat{Q}C$ (vertically opposite angles)

$\therefore \Delta APQ \equiv \Delta QCR$ (AAS)

$\therefore AP = RC$ and $PQ = QR$ (corresponding sides of congruent triangles)

However, $AP = PB$

$\therefore PB = RC$

In the quadrilateral $BCRP$, $PB = RC$ and $PB//RC.$

$\therefore BCRP$ is a parallelogram.

$\therefore PR = BC$ and $PR//BC.$

However, $PQ = QR.$

$\therefore PQ = \frac{1}{2} PR$

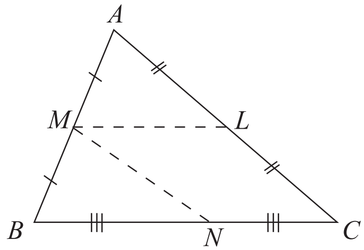
$= \frac{1}{2} BC$ (since $PR = BC$)

$\therefore PQ//BC$ and $PQ = \frac{1}{2} BC.$

Now let us consider how riders are proved using the midpoint theorem.

Example 1

M, N and *L* are the midpoints of the sides *AB, BC* and *CA* respectively of the triangle *ABC*. Prove that *NCLM* is a parallelogram.



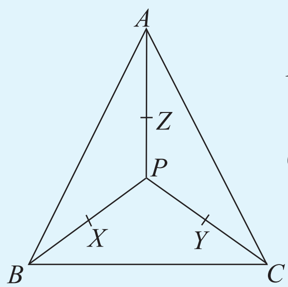
According to the midpoint theorem, $ML = \frac{1}{2} BC$

$= NC$ (since N is the midpoint of BC)

Further, $ML // BC$.

Therefore, a pair of opposite sides of the quadrilateral *NCLM*, are equal and parallel. Therefore, *NCLM* is a parallelogram.

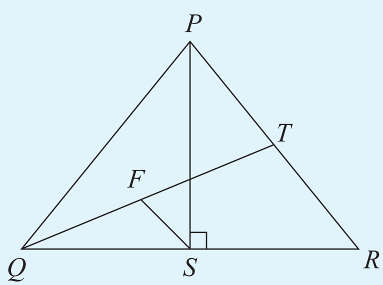
Exercise 11.2

1. 

P is a point inside the triangle *ABC*. The midpoints of *AP, BP* and *CP* are *Z, X* and *Y* respectively.

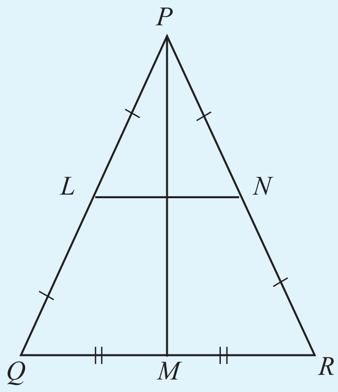
(i) Show that $B\hat{A}C = X\hat{Z}Y$, $A\hat{C}B = Z\hat{Y}X$ and $C\hat{B}A = Y\hat{X}Z$,

(ii) Show that the perimeter of triangle *ABC* is twice the perimeter of triangle *XYZ*.

2. 

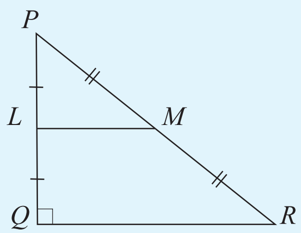
The bisector of the angle $Q\hat{P}R$ of the triangle *PQR* in the figure, meets the side *QR* at the point *S*, such that $PS \perp QR$. The midpoint of *QT* is *F*. Prove that $FS // TR$.

3.



Based on the information in the figure prove that $PM \perp LN$.

4.

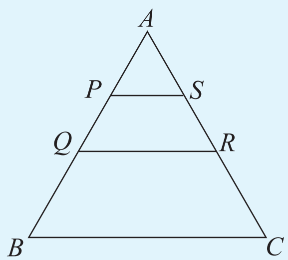


Based on the information in the figure prove that

(i) $\Delta\, PLM \equiv \Delta\, QLM$.

(ii) area of the quadrilateral $LQRM = \frac{3}{4}$area of $\Delta\, PQR$.

5.



The mid points of the sides AB and AC of the triangle ABC are Q and R respectively and the mid points of the sides AQ and AR of the triangle AQR are P and S respectively Prove that $4\, PS = BC$

6. (i) Prove that the quadrilateral that is formed by joining the midpoints of the sides of a quadrilateral is a parallelogram.

(ii) Prove that the quadrilateral that is formed by joining the midpoints of the sides of a rectangle is a rhombus.

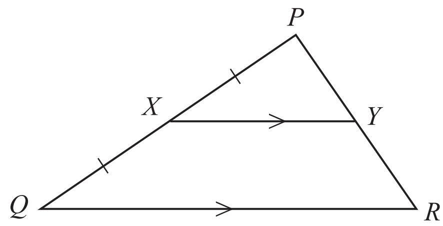
(iii) Prove that the quadrilateral that is formed by joining the midpoints of the sides of a square is a square.

(iv) Prove that the quadrilateral that is formed by joining the midpoints of the sides of a rhombus is a rectangle.

11.3 Converse of the Midpoint Theorem

Now let us consider the converse of the midpoint theorem.

Theorem:
The straight line through the midpoint of one side of a triangle and parallel to another side, bisects the third side.



X is the midpoint of the side PQ of the triangle PQR in the figure (that is, $PX = XQ$). If $XY//QR$, according to the converse of the midpoint theorem, Y is the midpoint of PR. That is, $PY = YR$.

Do the following activity to establish this theorem.

Activity 2

- Construct the triangle PQR such that $PQ = 5$ cm, $QR = 6$ cm and $RP = 7$ cm.
- Mark the midpoint of the side PQ as X.
- Name the point at which the straight line through X parallel to QR meets the side PR as Y.
- Measure the lengths of PY and YR and write down the relationship between these two lengths.
- Similarly, if the straight line through X parallel to PR meets the side QR at Z, write down the relationship between the lengths of QZ and ZR.

You would have observed by doing the above activity that $PY = YR$ and $QZ = ZR$. This establishes the fact that the straight line through the midpoint of one side of a triangle and parallel to another side, bisects the third side.

Now let us consider some applications of the converse of the midpoint theorem.

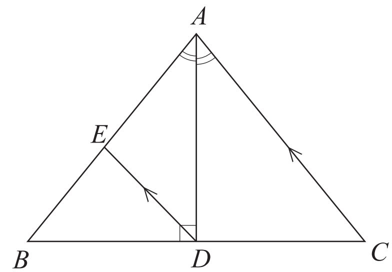
Example 1

The bisector of the angle $B\hat{A}C$ of the triangle ABC meets the side BC at D. $A\hat{D}B = 90^\circ$. The straight line through D parallel to CA meets the side AB at E.

Prove that

(i) $\Delta\ ADB \equiv \Delta\ ADC$,

(ii) $BE = EA$.



(i) In the triangles ADB and ADC,

$B\hat{A}D = C\hat{A}D$ (since AD is the bisector of $B\hat{A}C$)

AD is the common side

$A\hat{D}B = A\hat{D}C$ ($AD \perp BC$)

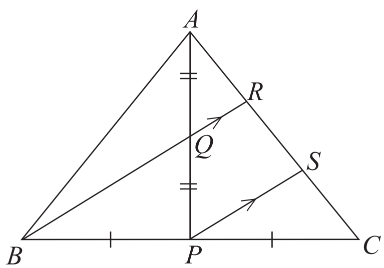
$\therefore \Delta\, ABD \equiv \Delta\, ADC$ (AAS)

(ii) $BD = DC$ (corresponding sides of the congruent triangles ADB and ADC)

Since $BD = DC$ and $AC \,//\, DE$, by the converse of the midpoint theorem,

$\underline{\underline{BE = EA}}$.

Example 2



P is the midpoint of the side BC of the triangle ABC in the figure. The midpoint of AP is Q. BQ produced meets the side AC at R. The line through P parallel to BR meets AC at S. If $AC = 15$ cm, find the length of AS.

In the triangle APS, $AQ = QP$ and $QR // PS$.

Therefore, according to the converse of the midpoint theorem,

$AR = RS$ ——— 1

In the triangle BRC, $BP = PC$ and $BR // PS$.

Therefore, according to the converse of the midpoint theorem,

$RS = SC$ ——— 2

From 1 and 2 we obtain, $AR = RS = SC$.

$$\therefore\ AS = \frac{2}{3} AC$$

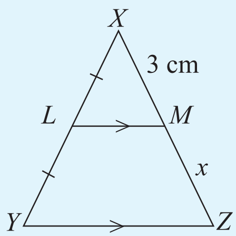
$$= \frac{2}{3} \times 15$$

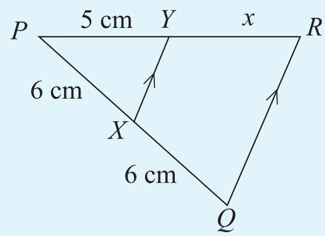
$$= 10.$$

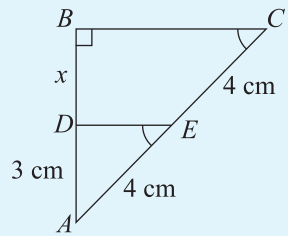
Therefore, the length of AS is 10 cm.

Exercise 11.3

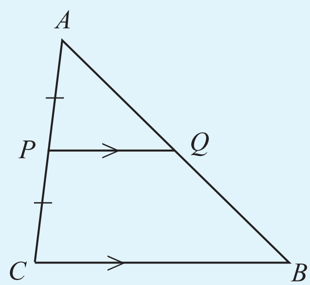
1. Find the value of x in each figure.







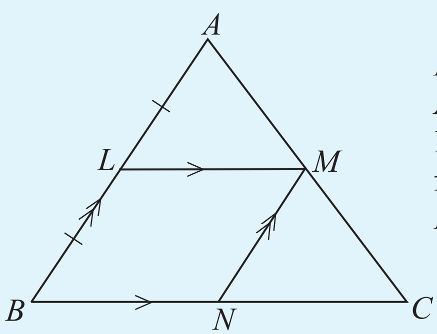
2.



The midpoint of the side AC of the triangle ABC is P.
$BC = 12$ cm, $AB = 15$ cm and $PQ // CB$.
Find

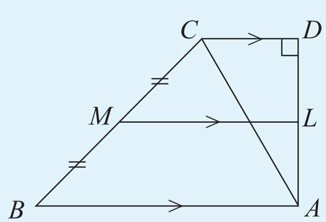
(i) the length of QB.
(ii) the length of PQ.

3.



L is the midpoint of the side AB of the triangle ABC in the figure. $LM//BC$ and $MN//AB$. If $AB = 10$ cm, $AM = 7$cm and $BC = 12$ cm, find the length of MC and the perimeter of $BNML$.

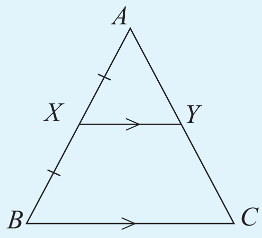
4.



Based on the information in the figure and if $AC = 10$ cm, $AD = 8$ cm and $ML = 10$ cm,

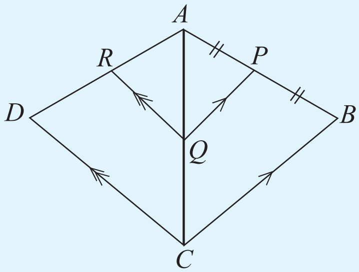
(i) find the length of DC.
(ii) find the area of the trapezium $ABCD$.

5.



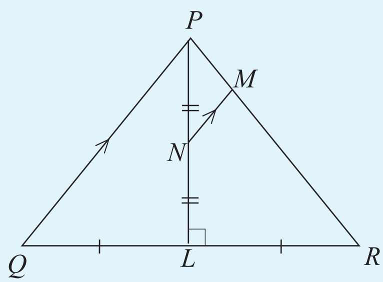
The perimeter of the equilateral triangle ABC in the figure is 30 cm. Based on the information in the figure find the perimeter of the trapezium $BCYX$.

6.



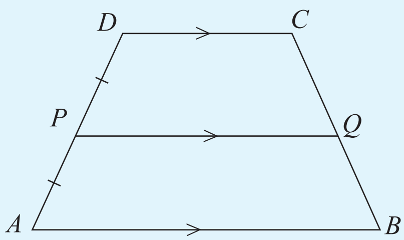
ABC and ADC in the figure are equilateral triangles and $AB = 20$ cm. Based on the information in the figure, find the perimeter of the region $PQRDCB$.

7.



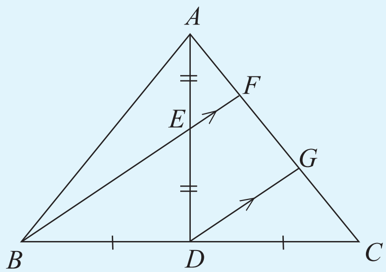
Find the length of MN based on the information in the figure, if $PQ = 20$ cm.

8.



Based on the information in the figure, express the length of PQ in terms of the lengths of AB and DC.

9.



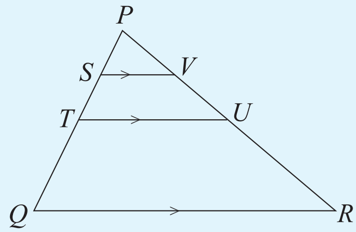
The length of a side of the equilateral triangle ABC is x cm. If $EF = y$ cm, based on the information in the figure, express the following in terms of x and y.

(i) The perimeter of $EDGF$.

(ii) The perimeter of $BDGF$.

(iii) The perimeter of $BDGA$.

10.

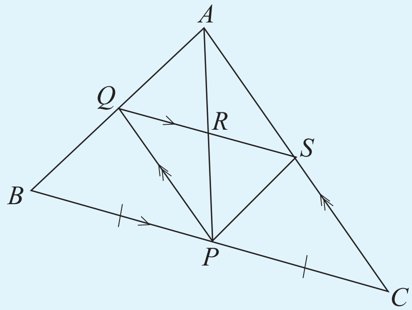


The midpoint of the side PQ of the triangle PQR in the figure is T. S is the midpoint of PT. The straight lines through S and T drawn parallel to QR, meet the side PR at V and U respectively.

(i) Prove that $PV = \frac{1}{4}PR$.

(ii) Determine the ratio $SV : QR$.

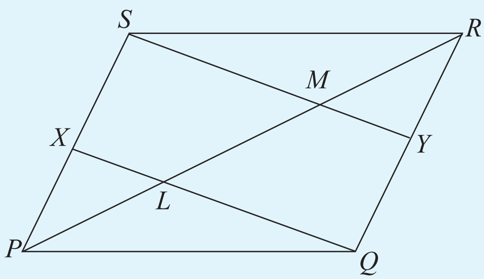
11.



Based on the information in the figure prove that $AR = RP$ and PS // BQ.

Miscellaneous Exercise

1.

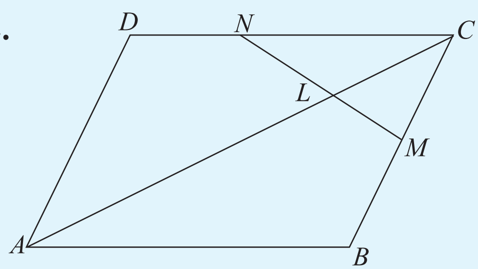


The midpoints of the sides PS and QR of the parallelogram $PQRS$ are X and Y respectively. The lines XQ and SY meet the diagonal PR at L and M respectively.

(i) Prove that $XQYS$ is a parallelogram.

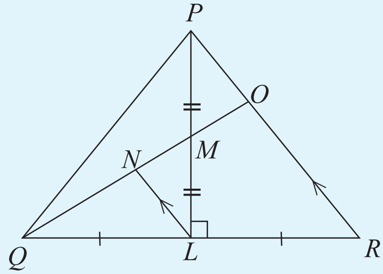
(ii) Prove that $PM = \frac{2}{3}PR$.

2.



The midpoints of the sides BC and CD of the parallelogram $ABCD$ are M and N respectively. The straight line MN intersects the diagonal AC at L.

Prove that $LC = \frac{1}{4}AC$.

3. 

Based on the information in the figure, prove the following.

(i) $QN = NO$.
(ii) $PNLO$ is a parallelogram.
(iii) $\Delta\, POM \equiv \Delta\, NLM$.
(iv) $MO = \frac{1}{4}\, QO$.

4. $PQRS$ is a parallelogram. Its diagonals intersect at O. The midpoint of the side PQ is L. The midpoint of LO is T. PT produced meets QR at Y. Prove that,

(i) $PT = TY$,
(ii) $PLYO$ is a parallelogram,
(iii) $4\, LT = QR$.

5. Y and X are the midpoints of the sides PQ and PR respectively of the triangle PQR. The lines QX and YR intersect each other at L. The straight line through Q parallel to YR meets PL produced at M. The lines LM and QR intersect at N.

(i) Prove that $PL = LM$.
(ii) Prove that $MR//QX$.
(iii) Prove that $QMRL$ is a parallelogram.
(iv) Determine $\frac{PL}{PN}$.

12 Graphs

By studying this lesson you will be able to

- find the solution to a pair of simultaneous equations graphically,
- sketch the graphs of quadratic functions of the form $y = ax^2 + bx + c$,
- analyze the behaviour of a function by considering its graph.

Do the following exercise to recall the facts that you have learnt earlier about straight lines and their graphs.

Review Exercise

1. ***a.*** On the same coordinate plane, draw the graphs of the following straight lines by calculating the y values corresponding to three selected values of x.

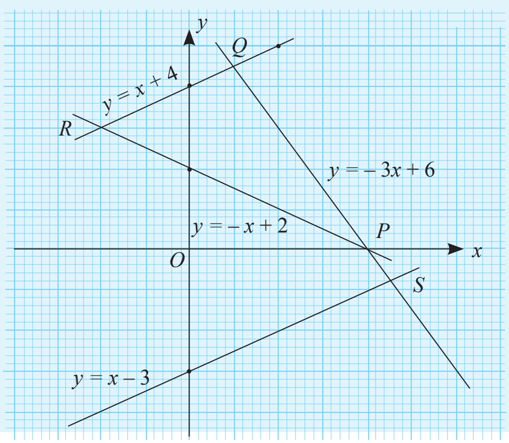
(i) $y = x + 1$ (ii) $y - x = 5$ (iii) $2y = -x - 4$ (iv) $3x + 2y = 6$

b. Write down the coordinates of the points at which each straight line intersects the main axes.

2. For each of the straight lines given below, determine whether the given pairs of coordinates lie on it or not.

(i) $y = 2x - 3$; $(1, 1), (0, 3), (2, 1)$ (ii) $y = 2x - 3$; $(0, -3), (\frac{1}{2}, 4), (1, 3)$

3. The graphs of four straight lines on a coordinate plane are given below. From the seven given pairs of coordinates, select the ones which correspond to the points P, Q, R and S, which are the intersecting points of the graphs. Give reasons for your answers.



$(-3, 5), (-1, 3), (-1, -3)$

$(\frac{1}{2}, 4\frac{1}{2}),\ (2, 0),\ (-\frac{5}{2}, \frac{3}{2}),$

$(2\frac{1}{4}, -\frac{3}{4})$

12.1 Finding the solution to a pair of simultaneous equations graphically

You have learnt in previous grades how to find the solution to a pair of simultaneous equations using algebraic methods. We will now consider how the solution to a pair of simultaneous equations is found by representing the equations graphically.

Let us consider the following pair of simultaneous equations.

$$y - x = -3$$
$$y + 3x = 5$$

Let us first solve this pair of simultaneous equations algebraically.

$$y - x = -3 \text{ ——— } (1)$$
$$y + 3x = 5 \text{ ——— } (2)$$

(2) – (1) gives us,

$$(y + 3x) - (y - x) = 5 - (-3)$$
$$\therefore\ y + 3x - y + x = 5 + 3$$
$$\therefore\ 4x = 8$$
$$\therefore\ x = 2$$

By substituting $x = 2$ in (1) we obtain,

$$y - x = -3$$
$$\therefore\ y = -3 + 2$$
$$\therefore\ y = -1$$

$\therefore$ the solution is

$x = 2$ and $y = -1$.

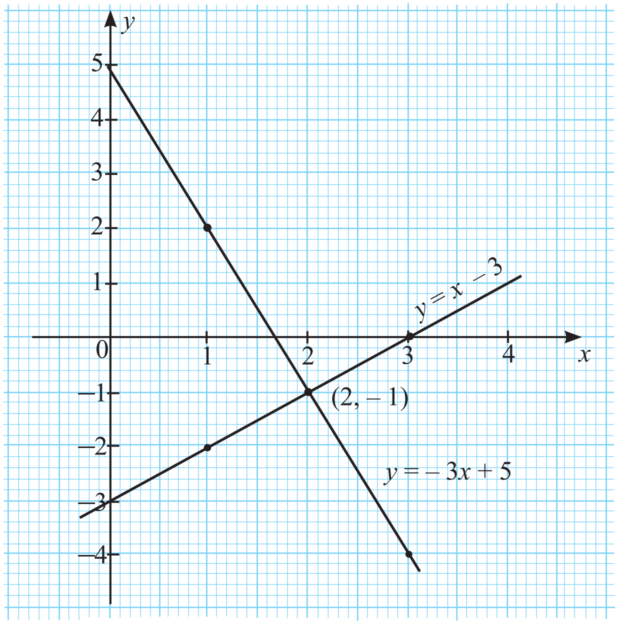
When these two equations are considered separately, they can be written as $y = x - 3$ and $y = -3x + 5$ by making y the subject. Let us first draw the graphs of these straight lines on the same coordinate plane. Two tables of values corresponding to these equations are given below.

$y = x - 3$

x	1	2	3
y	-2	-1	0

$y = -3x + 5$

x	1	2	3
y	2	-1	-4



The graphs of the straight lines drawn on the same coordinate plane using the above tables of values intersect each other at the point $(2, -1)$. When the x and y values corresponding to this point are substituted into the given pair of equations, it can be observed that the two sides of the equations are equal. That is, the coordinates $x = 2$ and $y = -1$ of the intersection point of these graphs is the solution to the given pair of simultaneous equations.

This geometric solution to the pair of simultaneous equations is further established by the fact that the same solution was obtained algebraically too.

Accordingly, to find the solution to a pair of simultaneous equations graphically, the straight lines corresponding to the pair of equations should be drawn on the same coordinate plane, and the coordinates of the point of intersection of the graphs should be found. Then the x and y values of the solution are obtained from the x-coordinate and the y-coordinate respectively of the point of intersection.

In the following example, the method of constructing a pair of simultaneous equations and solving it graphically is considered.

Example 1

A person buys 10 stamps, some of value Rs 10 and the rest of value Rs 20. The total value of the stamps is Rs 120.

(i) By taking the number of stamps of value Rs 10 that he bought as x and the number of stamps of value Rs 20 as y, construct a pair of simultaneous equations.

(ii) By solving the pair of simultaneous equations graphically, find the number of Rs 10 stamps and the number of Rs 20 stamps he bought.

The relevant pair of simultaneous equations is as follows.

$$x + y = 10 \text{ ——— } (1)$$
$$10x + 20y = 120 \text{ ——— } (2)$$

Let us represent each of the equations graphically.

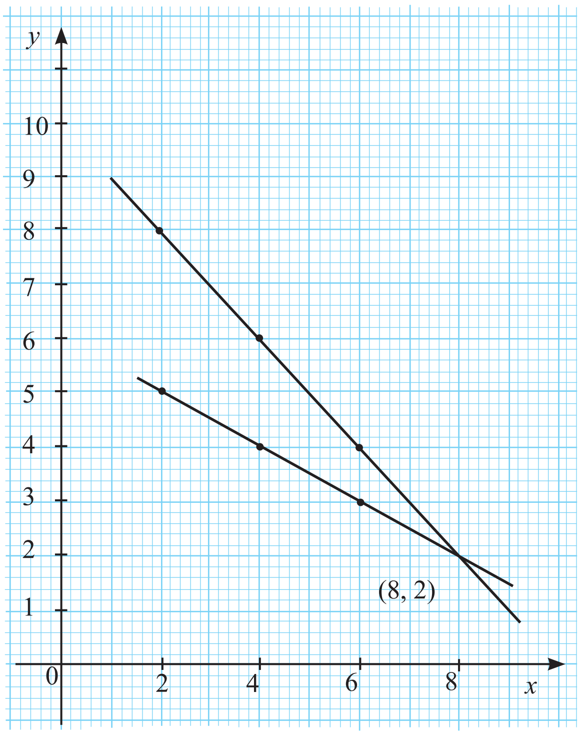
$x + y = 10$; that is, $y = -x + 10$

x	2	4	6
y	8	6	4

$10x + 20y = 120$; that is, $y = -\frac{1}{2}x + 6$

x	2	4	6
y	5	4	3

We obtain the following pair of straight lines.



When the pair of equations $x + y = 10$ and $10x + 20y = 120$ are represented graphically, they intersect each other at the point (8, 2). Therefore, the solution to the pair of equations is $x = 8$ and $y = 2$. That is, the person bought eight Rs 10 stamps and two Rs 20 stamps.

Exercise 12.1

1. Solve each of the following pairs of simultaneous equations graphically. Verify your answers by solving the equations algebraically too.

a. $y - x = 4$
$y - 2x = 3$

b. $y = -2x - 2$
$-2y = -x - 6$

c. $3x - 4y = 7$
$5x + 2y = 3$

2. A certain school has two Grade 11 classes A and B. If five students from class A move to class B, then the number of students in class B will be twice the number of students in class A. However, if five students from class B move to class A, then the number of students in the two classes will be equal.

(i) Construct a pair of simultaneous equations by taking the number of students initially in class A as x and the number of students initially in class B as y.

(ii) Draw the graphs of the pair of equations on the same coordinate plane and hence find the number of students in class A and in class B separately.

The graphs of quadratic functions

Do the following exercise to recall what you have learnt in Grade 10 regarding the graphs of quadratic functions of the form $y = ax^2$ and $y = ax^2 + b$.

Review Exercise

1. An incomplete table of x and y values prepared to sketch the graph of the function $y = x^2 - 5$ is given below.

x	-3	-2	-1	0	1	2	3
y	4	___	-4	-5	___	-1	4

a. (i) Complete the table.
(ii) Sketch the graph of the above function by selecting a suitable scale.

b. Using your graph, write down
(i) the minimum value of the function.
(ii) the coordinates of the minimum point.
(iii) the interval of values of x for which the function is negative.
(iv) the interval of values of x for which the function is increasing positively.
(v) the value of x when $y = -1$.

2. (i) Complete the following table to sketch the graph of the function $y = -2x^2 + 4$.

x	-3	-2	-1	0	1	2	3
y	-14	___	2	4	2	-4	-14

(ii) Sketch the graph of the function by selecting a suitable scale.

Using the graph,

(iii) write down the coordinates of the turning point of the graph.
(iv) find the x values for which the function takes the value 0.
(v) write down the interval of values of x for which the function is decreasing negatively.
(vi) Write down the interval of values of x for which $y \leq 2$.
(vii) Obtain an approximate value for $\sqrt{2}$, to the first decimal place.

3. Complete the following table without drawing the graphs of the given functions.

Function	Nature of the turning point (maximum/ minimum)	Equation of the axis of symmetry	Maximum/ minimum value	Coordinates of the turning point
(i) $y = 2x^2$	....	....	....	....
(ii) $y = \frac{1}{2}x^2$	....	....	....	....
(iii) $y = x^2 + 3$	....	....	....	....
(iv) $y = 1 - 2x^2$	Maximum	$x = 0$	1	(0, 1)
(v) $y = - 3x^2 - 4$	....	....	....	....
(vi) $y = \frac{3}{2}x^2 - 2$	....	....	....	....

12.2 The graph of a function of the form $y = ax^2 + bx + c$

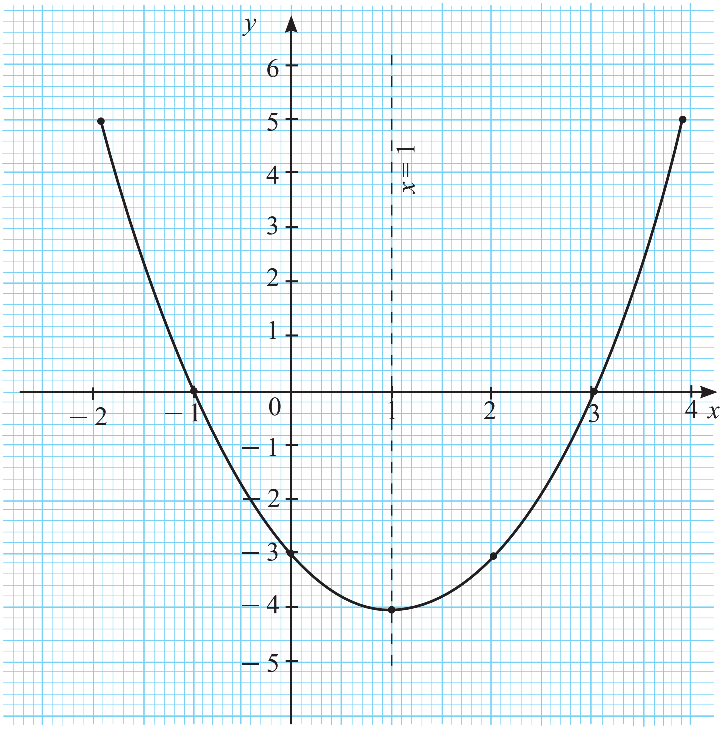
Let us first consider how we can use the knowledge we have gained on the characteristics of graphs of quadratic functions of the form $y = ax^2 + b$ to study the characteristics of graphs of quadratic functions of the form $y = ax^2 + bx + c$.

Drawing graphs of functions of the form $y = ax^2 + bx + c$ for $a > 0$ and identifying their characteristics

To identify some basic characteristics, let us first draw the graph of the function $y = x^2 - 2x - 3$. To do this, let us prepare a table as follows to obtain the values of y corresponding to the values of x for $-2 \leq x \leq 4$.

x	-2	-1	0	1	2	3	4
x^2	4	1	0	1	4	9	16
$-2x$	4	2	0	-2	-4	-6	-8
-3	-3	-3	-3	-3	-3	-3	-3
y	5	0	-3	-4	-3	0	5
(x, y)	$(-2, 5)$	$(-1, 0)$	$(0, -3)$	$(1, -4)$	$(2, -3)$	$(3, 0)$	$(4, 5)$

Before drawing the graph, it is important to consider the range of values that x and y take and prepare the coordinate plane accordingly. It is easy to draw the graph of $y = x^2 - 2x - 3$ by taking 10 small divisions along the x-axis to be one unit and 10 small divisions along the y-axis to be two units as scale.



The graph of a function of the form $y = ax^2 + bx + c$ is called a parabola.

We can identify the following characteristics by considering the graph.

- The graph is symmetric about the line $x = 1$. Therefore, the equation of the axis of symmetry of the graph is $x = 1$.

As the value of x increases gradually from – 2, the corresponding y value decreases gradually until it attains its minimum value – 4 and then starts increasing again.

Let us describe the behaviour of y further for the given range of values of x.

- As the value of x increases from – 2 to – 1, the value of y, that is, the value of the function decreases positively from 5 to 0. What is meant by "decreases positively" is that the function decreases while remaining positive.
- The function takes the value 0 when $x = -1$.
- As the value of x increases from – 1 to 1, the corresponding y value decreases negatively from 0 to – 4.
- As the value of x increases from 1 to 3, the corresponding y value increases negatively from – 4 to 0 .
- The function takes the value 0 when $x = 3$.
- As the value of x increases from 3, the value of y increases positively.

By considering the above characteristics, the range of values of x for which the function is negative can be expressed in terms of inequalities as $-1 < x < 3$.

- The value of y is positive when the value of x is less than -1 or greater than 3. That is, the range of values of x for which the function is positive is $x < -1$ and $x > 3$.

Let us also pay attention to the following facts.

- It is very important to understand the relationship between the graph that has been drawn and the function $y = x^2 - 2x - 3$.

This can be described as follows.

(1) If any point (a, b) lies on the graph, then the equation $y = x^2 - 2x - 3$ is satisfied by $x = a$ and $y = b$. That is the equation $b = a^2 - 2a - 3$ is true.

(2) Conversely, if $x = a$ and $y = b$ satisfies the equation $y = x^2 - 2x - 3$, then the point (a, b) lies on the graph of the function $y = x^2 - 2x - 3$.

It is extremely important to keep the above two facts in mind. Clearly the point $(-1, 0)$ lies on the graph. Therefore, $x = -1$ and $y = 0$ should satisfy the equation $y = x^2 - 2x - 3$. That is, $0 = (-1)^2 - 2(-1) - 3$. This can be verified by simplifying the right hand side. Another way of stating this is that $x = -1$ is a root of the equation $x^2 - 2x - 3 = 0$. By the same argument it can be stated that $x = 3$ is also a root of this equation. That is, the roots of the equation $x^2 - 2x - 3 = 0$ are the x coordinates of the points at which the graph of $y = x^2 - 2x - 3$ intersects the x- axis. This can be written more generally as follows.
The x coordinates of the points at which the graph of $y = ax^2 + bx + c$ intersects the x - axis are the roots of the quadratic equation $ax^2 + bx + c = 0$.

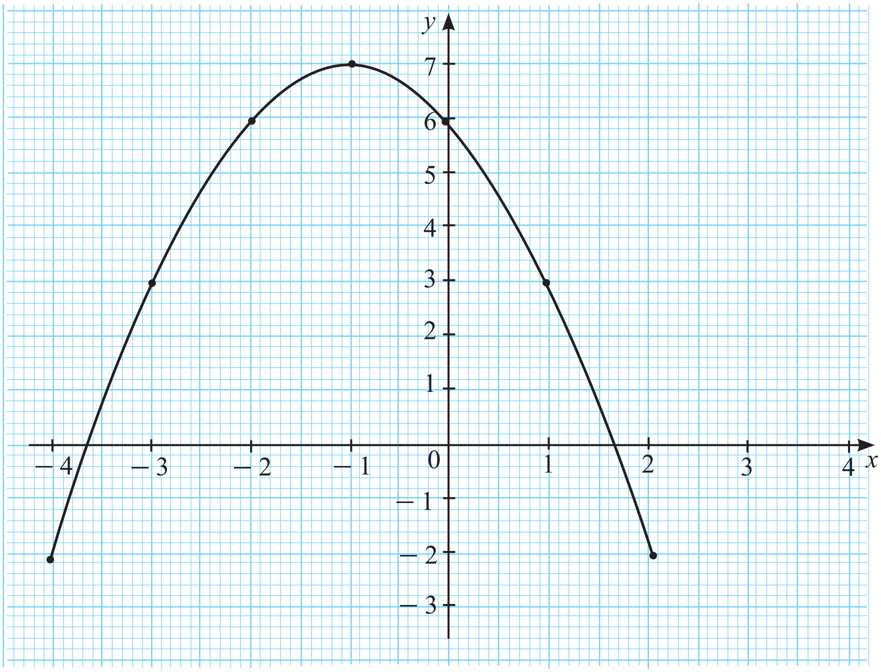
- The minimum value of the function is obtained at the turning point of the graph. The minimum value is -4. The coordinates of the minimum point are $(1, -4)$.

Drawing the graphs of functions of the form $y = ax^2 + bx + c$ for $a < 0$ and identifying their characteristics

Let us prepare a table as follows for $-4 \leq x \leq 2$ to draw the graph of the function $y = -x^2 - 2x + 6$.

x	-4	-3	-2	-1	0	1	2
$-x^2$	-16	-9	-4	-1	0	-1	-4
$-2x$	8	6	4	2	0	-2	-4
$+6$	$+6$	$+6$	$+6$	$+6$	$+6$	$+6$	$+6$
y	-2	3	6	7	6	3	-2
(x, y)	$(-4, -2)$	$(-3, 3)$	$(-2, 6)$	$(-1, 7)$	$(0, 6)$	$(1, 3)$	$(2, -2)$

By considering the range of values of x and y, and selecting 10 small divisions along the x-axis to be one unit and 10 small division along the y axis to be two units as scale, the graph of the given function can be drawn as follows.



We can identify the following characteristics by considering the graph.

- The graph which has a maximum value 7 is symmetric about the line $x = -1$.
- Therefore, the equation of the axis of symmetry of the graph is $x = -1$.
- The coordinates of the turning point are $(-1, 7)$.
- As the value of x increases from -4 to -3.6, the value of y increases negatively.
- The function takes the value 0 when $x = -3.6$.
- As the value of x increases from -3.6 to -1, the corresponding y value increases positively from 0 to 7.
- The function attains its maximum value of $+7$ when the value of x is -1.
- As the value of x increases from -1 to $+1.6$, the value of the function decreases positively.
- The function takes the value 0 when $x = +1.6$.
- As the value of x increases from $+1.6$, the value of y decreases negatively.
- When the value of x lies between -3.6 and $+1.6$, the value of the function is positive. That is, the range of values of x for which the function is positive is $-3.6 < x < +1.6$.
- When the value of x is less than -3.6 or greater than $+1.6$, the function is negative.
- That is, the range of values of x for which the function is negative is $x < -3.6$ and $x > +1.6$.

- The graph intersects the line $y = 0$ (x-axis) when $x = -3.6$ and $x = +1.6$. Therefore, the values of x which satisfy the equation $-x^2 - 2x + 6 = 0$, that is, the roots of this equation are $x = -3.6$ and $x = +1.6$.
- The maximum and minimum values that the function takes when $0 \leq x \leq 2$, are 6 and -2 respectively.

Exercise 12.2

1. Draw the graph of the function in the given range by selecting a suitable scale.

$y = x^2 + 2x - 7 \qquad (-4 \leq x \leq 2)$

Using your graph, write down,

(a) the minimum value of the function
(b) the coordinates of the turning point,
(c) the equation of the axis of symmetry after drawing it,
(d) the values of x for which $y = 0$,
(e) the range of values of x for which the function is negative,
(f) the range of values of x for which the function is positive,
(g) the range of values of x for which the function is decreasing positively,
(h) the range of values of x for which the function is increasing negatively.

2. An incomplete table of values prepared to sketch the graph of $y = x^2 - 4x + 2$ is given below.

x	-1	0	1	2	3	4	5
y	____	2	-1	____	-1	2	7

(i) Complete the above table and by taking 10 small divisions along the x-axis and y-axis to be one unit as scale, draw the graph of the given function.

(ii) By considering the graph, write down,
(a) the coordinates of the turning point,
(b) the minimum value of the function,
(c) the values of x for which the function is zero,
(d) the range of values of x for which $y \leq -1$,
(e) the roots of the equation $x^2 - 4x + 2 = 0$,

3. For the following function, draw the graph in the given range by selecting a suitable scale.

$y = -x^2 - 2x + 3 \ (-4 \leq x \leq 2)$

Using your graph write down,
(a) the maximum value of the function,
(b) the coordinates of the turning point,

(c) the equation of the axis of symmetry after drawing it,
(d) the values of x for which $y = 0$,
(e) the range of values of x for which the function is positive,
(f) the range of values of x for which the function is negative,
(g) the range of values of x for which the function is increasing positively,
(h) the range of values of x for which the function is decreasing negatively.

4. An incomplete table of values of x and y prepared to draw the graph of the function $y = -2x^2 + 3x + 2$ is given below.

x	-2	-1	0	$\frac{3}{4}$	1	2	3	3.5
y	-12	-3	2	___	3	___	-7	-12

(i) Complete the above table and by taking 10 small divisions along the x-axis and y-axis to be one unit as scale, draw the graph of the given function.

(ii) By considering the graph, write down
(a) the coordinates of the turning point,
(b) the equation of the axis of symmetry,
(c) the roots of the equation $-2x^2 + 3x + 2 = 0$,
(d) the range of values of x for which the function is increasing positively,
(e) the value of x for which the value of the function is 4,
(f) the values of x for which y is -4.

12.3 The graph of a function of the form $y = \pm (x + b)^2 + c$.

$y = \pm (x + b)^2 + c$ is also an equation of a quadratic function. Here the quadratic function has been expressed in the special form $y = \pm (x + b)^2 + c$. When it is written in this form, some of the characteristics of its graph can be inferred without drawing the graph. Several such characteristic are given in the following table.

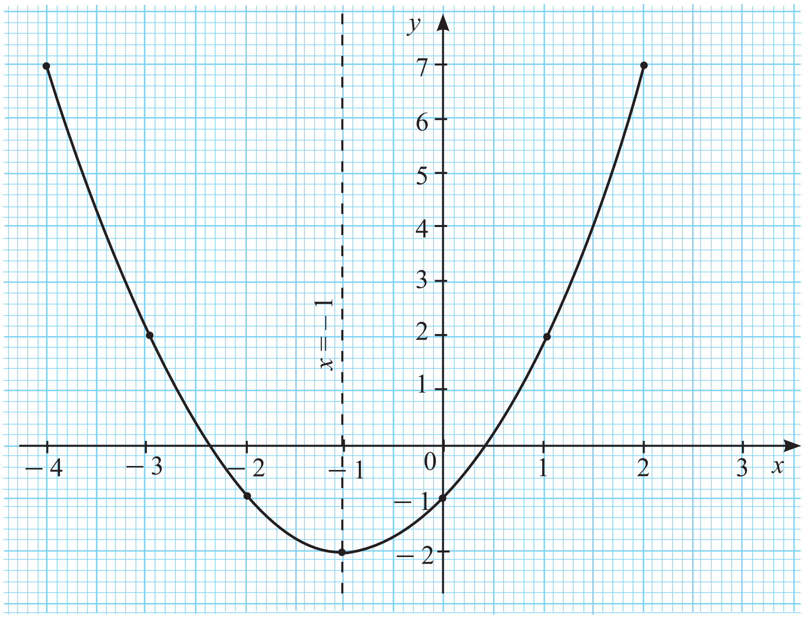
Equation of the function	Nature of the turning point	Maximum/ minimum value of the function	Coordinates of the maximum/ minimum point of the graph	Equation of the axis of symmetry of the graph	The point at which the function intersects the y-axis
$y = (x + b)^2 + c$	Minimum	c	$(-b, c)$	$x = -b$	$(0, b^2 + c)$
$y = -(x + b)^2 + c$	Maximum	c	$(-b, c)$	$x = -b$	$(0, -b^2 + c)$

To verify the characteristics in the above table, let us consider the following example.

Let us consider the function $y = (x + 1)^2 - 2$. This is of the form $y = (x + b)^2 + c$ where $b = 1$ and $c = -2$. To draw the graph of this function for the values of x from -4 to $+2$, let us calculate the necessary values of y as shown in the following table.

x	-4	-3	-2	-1	0	1	2
$(x+1)^2$	9	4	1	0	1	4	9
-2	-2	-2	-2	-2	-2	-2	-2
y	7	2	-1	-2	-1	2	7
(x, y)	$(-4, 7)$	$(-3, 2)$	$(-2, -1)$	$(-1, -2)$	$(0, -1)$	$(1, 2)$	$(2, 7)$

By taking 10 small divisions along the x-axis to be one unit and 10 small divisions along the y-axis to be two units as scale, the graph of the given function can be drawn as follows.



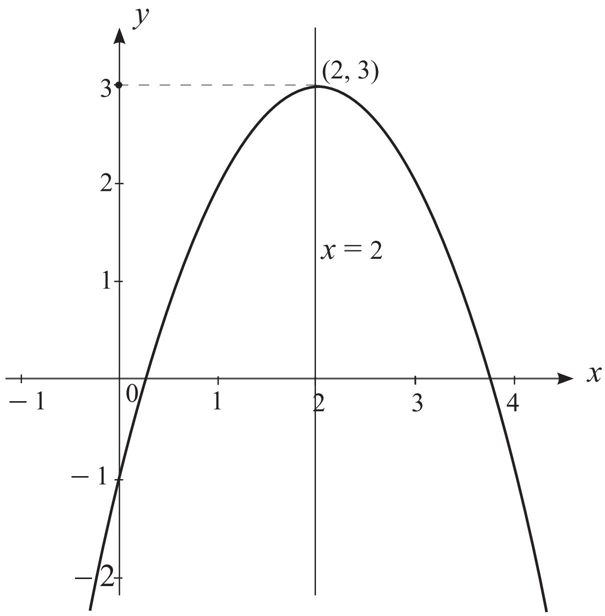
Note:
This graph has a minimum point. The minimum value of the function is -2 $(= c)$. The coordinates of the minimum point are $(-1, -2)$, that is, $(-b, c)$. The axis of symmetry is $x = -1$ (That is, $x = -b$).

When a quadratic function has been given in the form $x = \pm(x + b)^2 - c$, the graph can be sketched using the characteristics in the above table. The example given below explains how this is done.

Example 1

Sketch the graph of $y = -\ (x-2)^2 + 3$.

Since the coefficient of $(x-2)^2$ is negative, the turning point of the graph is a maximum. The coordinates of this maximum point is $(2, 3)$. The axis of symmetry of the graph is $x = 2$. Further, to find the point at which the graph intersects the y-axis, let us substitute $x = 0$ in $y = -\ (x-2)^2 + 3$. We then obtain $y = -(0-2)^2 + 3 = -1$. Accordingly, we can sketch the graph as follows.



Example 2

Write down the following for the function $y = x^2 + 3x - 4$.

(i) The nature of the graph
(ii) The equation of the axis of symmetry
(iii) The maximum/minimum value
(iv) The coordinates of the turning point of the graph

The function has been given in the form $y = ax^2 + bx + c$.
Let us first write it in the form $y = (x + k)^2 + h$.

$$y = x^2 + 3x - 4$$

$$y = (x + \frac{3}{2})^2 - 4 - \frac{9}{4}$$

$y = (x + \frac{3}{2})^2 - \frac{25}{4}$

(i) A parabola with a minimum.

(ii) $x = -\frac{3}{2}$. That is, $x = -1\frac{1}{2}$.

(iii) Minimum value is $-\frac{25}{4}$.

(iv) $(-\frac{3}{2}, -\frac{25}{4})$

Exercise 12.3

1. Sketch the graph of each of the following functions in the given interval by selecting a suitable scale.

(i) $y = (x - 2)^2 - 3 \quad (-1 \leq x \leq 5)$ (ii) $y = (x + 3)^2 - 4 \quad (-6 \leq x \leq 0)$

By considering each graph, write down

a. the minimum value of the function,
b. the coordinates of the minimum point of the graph,
c. the equation of the axis of symmetry after drawing it,
d. the range of values of x for which the function is positive,
e. the values of x for which $y = 0$,
f. the range of values of x for which the function is negative.

2. Sketch the graph of each of the following functions in the given interval by selecting a suitable scale.

(i) $y = -(x + 2)^2 + 2 \quad (-5 \leq x \leq 1)$ (ii) $y = -(x - 1)^2 + 3 \quad (-2 \leq x \leq 4)$

By considering each graph, write down

a. the maximum value of the function,
b. the coordinates of the maximum point of the graph,
c. the equation of the axis of symmetry after drawing it,
d. the range of values of x for which the function is positive,
e. the range of values of x for which the function is negative,
f. the values of x for which $y = 0$,
g. the range of values of x for which the function is increasing positively,
h. the range of values of x for which the function is decreasing negatively.

3. Draw a rough sketch of the graph of each of the functions.

(i) $y = (x - 2)^2 - 3$

(ii) $y = 2 - (x + 5)^2$

(iii) $y = x^2 + 6x - 1$

4. Without sketching the graph, write down the following for each of the given functions.

a. Nature of the graph
b. Equation of the axis of symmetry
c. Maximum/minimum value
d. Coordinates of the turning point

(i) $y = (x + 2)^2 - 3$ (ii) $y = -(x - 2)^2 + 4$ (iii) $y = -(x - \frac{3}{2})^2 + 1$

(iv) $y = 1\frac{1}{2} - (x - \frac{1}{2})^2$ (v) $y = 3\frac{1}{3} + (x + 2\frac{1}{2})^2$ (vi) $y = x^2 + 6x + 5$

12.4 Graphs of functions of the form $y = \pm (x \pm a)(x \pm b)$

$y = \pm (x + a)(x + b)$ is also an equation of a quadratic function. The function has been written in the special form $y = x \pm (x + a)(x + b)$. As in the above case, when the function is given in this form, certain characteristics of the graph of the function can be inferred without sketching the graph. The following table provides several such characteristics.

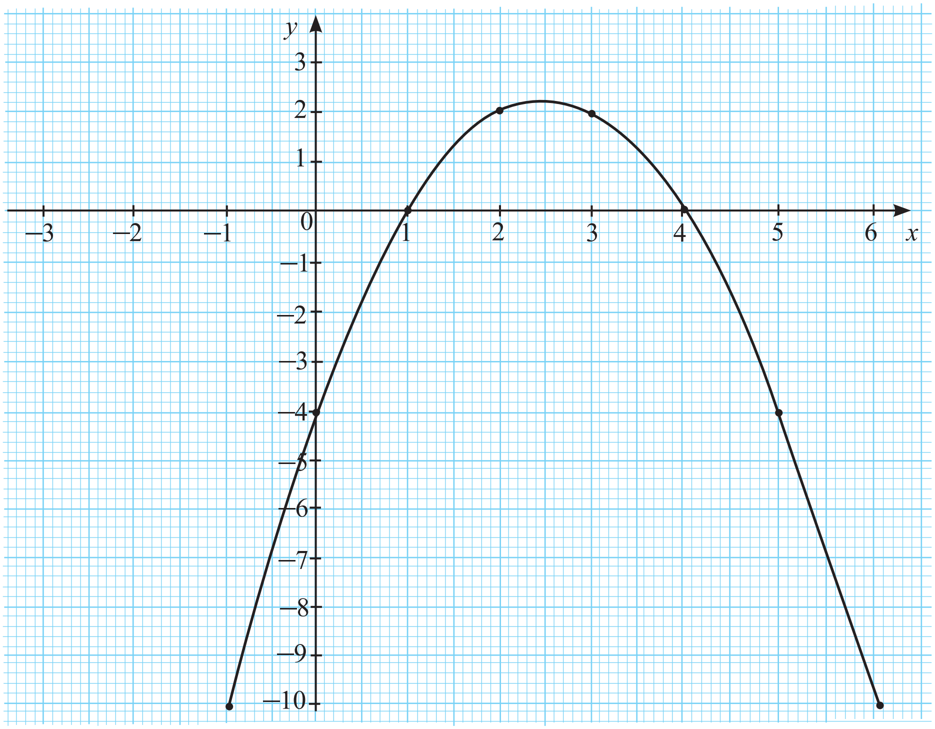
Equation of the function	Nature of the turning point	The coordinates of the maximum/ minimum point	Equation of the axis of symmetry of the graph	The points at which the function intersects the x-axis	The point at which the function intersects the y-axis
$y = (x + a)(x + b)$	Minimum	$\left(-\frac{(a+b)}{2}, -\frac{(a-b)^2}{4}\right)$	$x = -\frac{(a+b)}{2}$	$(-a, 0)$ and $(-b, 0)$	$(0, ab)$
$y = -(x + a)(x + b)$	Maximum	$\left(-\frac{(a+b)}{2}, \frac{(a-b)^2}{4}\right)$	$x = -\frac{(a+b)}{2}$	$(-a, 0)$ and $(-b, 0)$	$(0, -ab)$

To verify the characteristics in the above table let us consider the following example.

Let us consider $y = -(x - 1)(x - 4)$. This is of the form $y = -(x + a)(x + b)$ where $a = -1$ and $b = -4$. To draw the graph of this function, let us prepare a table of (x, y) values as follows.

x	-1	0	1	2	3	4	5	6
$-(x-1)(x-4)$	-10	-4	0	2	2	0	-4	-10
(x, y)	$(-1, -10)$	$(0, -4)$	$(1, 0)$	$(2, 2)$	$(3, 2)$	$(4, 0)$	$(5, -4)$	$(6, -10)$

By taking 10 small divisions along the x-axis to be one unit and 10 small divisions along the y-axis to be two units, as scale, the graph of the given function can be drawn as follows.



Verify as in the example of section 12.3 that this graph has the characteristics given in the table.

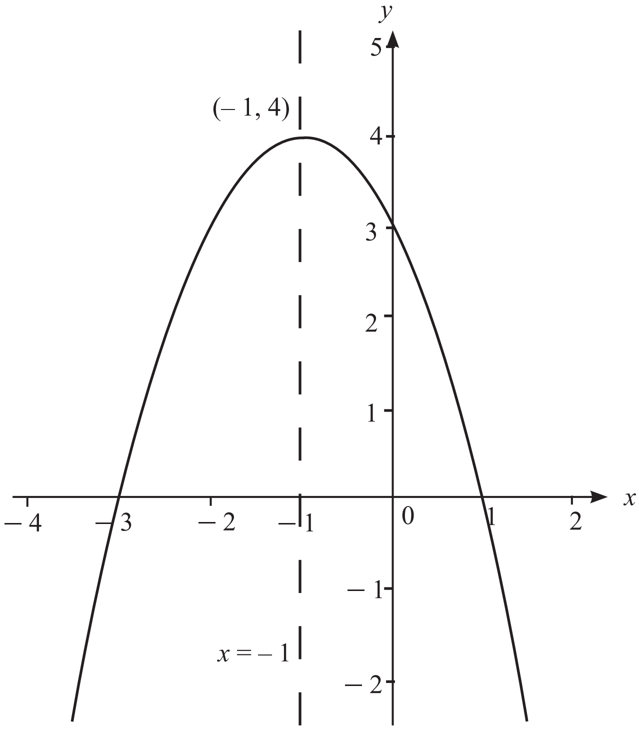
When a quadratic function is given in the form $y = \pm (x + a)(x + b)$, its graph can be sketched by considering the characteristics given in the table.

The following example describes how this is done.

Example 1

Sketch the graph of $y = -(x + 3)(x - 1)$.

This is of the form $y = -(x + a)(x + b)$, where $a = 3$ and $b = -1$ Since the coefficient of x^2 is negative, the turning point is a maximum. The points at which the graph intersects the x-axis are $(-3, 0)$ and $(1, 0)$. The coordinates of the maximum point are $\left(-\dfrac{a+b}{2}, \dfrac{+(a-b)^2}{4}\right) = (-1,\ 4)$. Accordingly, a graph of the following form can be drawn.



Example 2

Without drawing the graph of the function $y = x^2 + 5x - 14$, write down the following.

(i) The nature of the graph
(ii) The equation of the axis of symmetry
(iii) The maximum/minimum value
(iv) The coordinates of the turning point
(v) The coordinates of the points at which the graph intersects the x-axis

Let us organize this function in the form $y = (x + a)(x + b)$.
The given function can be written as $y = (x - 2)(x + 7)$.

(i) The graph is a parabola with a minimum.
(ii) Since $a = -2$ and $b = 7$, the axis of symmetry is

$$x = \frac{-(a+b)}{2} = \frac{-(-2+7)}{2};\ \text{that is,}\ x = -\frac{5}{2}.$$

(iii) Since the minimum value is obtained from $\frac{-(a-b)^2}{4}$,

the minimum value is $\frac{-(-2-7)^2}{4} = -\frac{81}{4}$.

(iv) The coordinates of the minimum point are $(-\frac{5}{2}, -\frac{81}{4})$.

(v) The coordinates of the intersection point of the graph and the x-axis are given by $(-a, 0)$ and $(-b, 0)$; that is, $(2, 0)$ and $(-7, 0)$.

Exercise 12.4

1. Sketch the graph of each of the given functions for the given range of values of x by selecting a suitable scale.

(a) $y = (x+1)(x+6)$ $\quad(-7 \leq x \leq 0)$
(b) $y = (x-2)(x-5)$ $\quad(0 \leq x \leq 7)$
(c) $y = -(x+1)(x+3)$ $\quad(-5 \leq x \leq 1)$
(d) $y = -(x-5)(x-3)$ $\quad(+1 \leq x \leq 7)$

For each of the graphs, write down the following.
(i) The values of x for which y is zero.
(ii) The equation of the axis of symmetry after drawing it.
(iii) The minimum/maximum value of the function.
(iv) The coordinates of the minimum/maximum point.
(v) The range of values of x for which the function is positive.
(vi) The range of values of x for which the function is negative.
(vii) The behaviour of y as x increases within the given range.

2. Draw a rough sketch of the graph of each of the functions.

(i) $y = (x-3)(x+5)$
(ii) $y = (x-1)(x-2)$
(iii) $y = -(x+3)(x-6)$

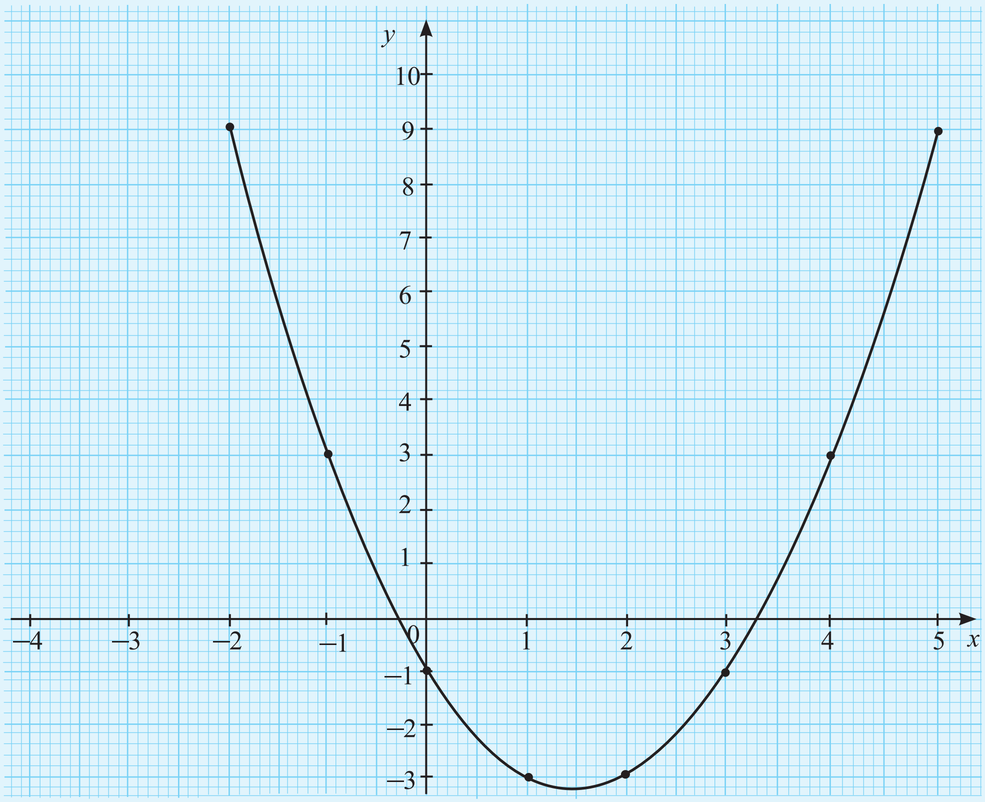
3. Without sketching the graph, write down the following for each of the given functions.

a. Nature of the graph **b.** Equation of the axis of symmetry
c. Maximum/minimum value **d.** Coordinates of the turning point.

(i) $y = (x-2)(x+3)$ (ii) $y = (x+1)(x-4)$ (iii) $y = (x-4)(x-1)$
(iv) $y = -(x-\frac{1}{2})(x+3)$ (v) $y = x^2 - 1\frac{1}{2}x - 2\frac{1}{2}$ (vi) $y = x^2 - 4x + 7$
(vii) $y = -x^2 - 6x - 5$ (viii) $y = -x^2 + 12x + 35$ (ix) $y = x^2 - x + 4$

Miscellaneous Exercise

1. (*a*) A sketch of the graph of a quadratic function for values of x such that $-2 \leq x \leq 5$ is given below.



Answer the following by considering this graph.

(i) Find the value of y when $x = 3$.

(ii) Draw the axis of symmetry and write down its equation.

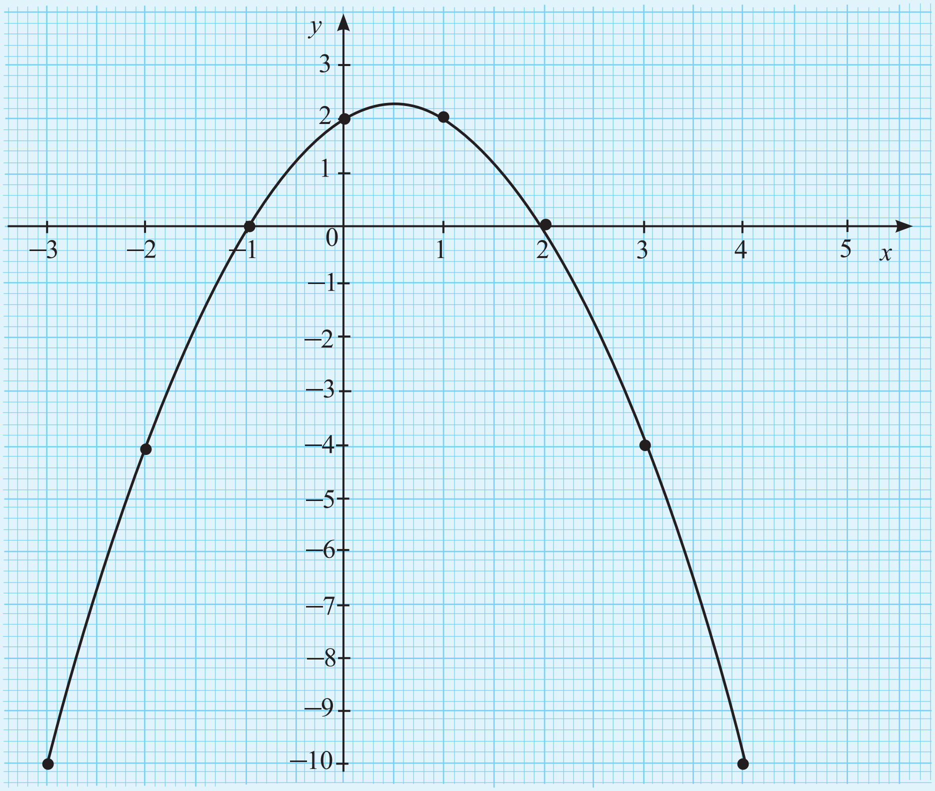
(iii) Write down the interval of values of x for which the function is negative.

(iv) This function can be expressed in the form $y = (x - a)^2 + b$. Determine the values of a and b.

(v) For the function obtained in (iv), find the values of x for which $y = 0$.

(vi) Write down the function which has the same axis of symmetry as in (ii), a maximum value of 5 and coefficient of x^2 equal to 1.

(*b*) A sketch of the graph of a quadratic function for values of x such that $-3 \leq x \leq 4$ is given below.



(i) Write down the values of x for which $y = 0$.

(ii) The relevant function can be expressed in the form $y = -(x - a)(x - b)$. By considering the answer to (i) above, determine the values of a and b.

(iii) Express the equation of the function obtained in (ii) above by substituting the values of a and b, in the form $y = -(x - p)^2 + q$ and obtain the coordinates of the maximum point. Verify your answer by considering the graph.

(iv) Write down the interval of values of x for which $y \leq -4$.

(v) Write down the interval of values of x for which the function is increasing positively.

2. $(x + 2)$ and $(3 - x)$ represent two numbers. $y = (x + 2)(3 - x)$ denotes the product of these two numbers.

(i) Complete the following table.

x	-3	-2	-1	0	1	2	3	4
y	-6	___	___	6	___	4	___	-6

(ii) Sketch the graph of the function y by selecting a suitable scale.

Answer the following questions by considering the graph.

(iii) Find the maximum value of the product.

For free distribution

(iv) Find the value of x for which the product is maximum.
(v) Write down the values of x for which the product is zero.
(vi) Write down the interval of values of x for which $y > 3$.
(vi) For which interval of values of x does the product increase gradually?
(viii) For which interval of values of x is the product positive?
(ix) Write down the maximum and minimum values of the product for $-1 \leq x \leq 3$.
(ix) Write down the maximum and minimum values of the product for $5 \leq x \leq 8$.

3. An incomplete table of several values of x and corresponding values of the function $y = (x-2)^2 - 2$ are given below.

x	-1	0	1	2	3	4	5
y	7	2	-1	-2	___	2	7

(i) Find the value of y when $x = -2$.
(ii) Sketch the graph of the given function by selecting a suitable scale.
(iii) Write down the coordinates of the turning point.
(iv) Write down the interval of values of x for which $y < 0$.
(v) By using the graph and algebraically, find the roots of the equation $x^2 - 4x + 2 = 0$ and hence obtain an approximate value for $\sqrt{2}$.
(vi) Write down the values of x for which the value of the function is 3.

4. An incomplete table of x and y values suitable to sketch the graph of $y = -(x+1)(x-3)$ is given below.

x	-2	-1	0	1	2	3	4
y	___	0	3	4	3	___	-5

(i) Find the value of y when $x = -2$ and $x = 3$.
(ii) Sketch the graph of the given function by selecting a suitable scale.
(iii) Write down the coordinates of the maximum point.
(iv) Obtain the values of x for which $y = 0$, and thereby verify that the maximum value is correct.
(v) Write down the interval of values of x for which $y \geq -1$.
(vi) Write down the roots of the equation $-x^2 + 2x + 3 = 0$.
(vii) Describe the behavior of the function for $1 \leq x \leq 4$.

5. An incomplete table of x and y values suitable to sketch the graph of $y = 5 - x - x^2$ is given below.

x	-4	-3	-2	-1	0	1	2	3
y	___	-1	3	5	5	___	-1	-7

(i) Find the value of y when $x = -4$ and $x = 1$.

(ii) Sketch the graph of the given function by selecting a suitable scale.

(iii) Write down the coordinates of the maximum point.

(iv) Write down the interval of values of x in which the function increases from -5 to $+3$.

(v) Write down the interval of values of x for which the function is negative.

(vi) Write down the roots of the equation $-x^2 - x + 5 = 0$ by considering the graph.

(vii) Deduce the coordinates of the maximum point of the function given by $y - 3 = 5 - x - x^2$.

13 Equations

By studying this lesson you will be able to:

- construct simultaneous equations with rational coefficients,
- solve simultaneous equations with rational coefficients,
- solve quadratic equations by means of factoring, completing the square and using the formula.

Solving Simultaneous Equations

To review what you have learnt so far on solving simultaneous equations, do the following problems.

Review Exercise

1. Solve the following pairs of simultaneous equations.

a. $6x + 2y = 1$
$4x - y = 3$

b. $a + 2b = 3$
$2a + 3b = 4$

c. $m - 4n = 6$
$3m + 2n = 4$

d. $9p - 2q = 13$
$7p - 3q = 0$

e. $2x + 3y = 12$
$3x - 4y = 1$

f. $3a + 12 = 2b$
$13 + 2a = 3b$

2. Sarath has twenty, two rupee and five rupee coins, which total Rs 55. Let x be the number of two rupee coins and y be the number of five rupee coins Sarath has.
(i) Express the given information in two equations.
(ii) How many of each type of coin does he have?

3. Malini and Nalini have a certain amount of money. When you add Rs 30 to the sum of the amounts Malini and Nalini have, the total amounts to Rs 175. Nalini has Rs 95 less than twice the amount Malini has. Let Rs x be the amount of money Malini has and Rs y be the amount of money Nalini has,
(i) Express the given information in two equations.
(ii) How much does each person actually have?

4. "It costs Rs 65 to buy two books and a pen. You can buy one such book at the cost of two such pens." Construct two simultaneous equations to represent this information and solve them to find the price of a book and a pen separately.

13.1 Simultaneous equations with fractional coefficients

We have learnt before how to solve a pair of simultaneous equations when the unknowns have integers as coefficients. Now we will explore through an example, how to solve a pair of simultaneous equations when the unknowns have fractions as coefficients.

Example 1

Kamal and Nimal have a certain amount of money. When you add $\frac{1}{2}$ of the amount Kamal has to $\frac{1}{3}$ of the amount Nimal has to get Rs 20. If $\frac{1}{4}$ th of what Kamal has is equal to $\frac{1}{6}$ th of what Nimal has, find the amounts Nimal and Kamal have separately.

Let us see how we can construct a pair of simultaneous equations to solve this problem.

Let us take the amount Kamal has as Rs x and the amount Nimal has as Rs y. Then, when you add $\frac{1}{2}$ of what Kamal has, that is, Rs $\frac{1}{2}x$ to $\frac{1}{3}$ of what Nimal has, that is, $\frac{1}{3}y$, you get $\frac{1}{2}x + \frac{1}{3}y$. Since this is equal to Rs 20 we get

$$\frac{1}{2}x + \frac{1}{3}y = 20. \text{ ——— } ① \text{ as one equation.}$$

Similarly, since $\frac{1}{4}$ th of what Kamal has is equal to $\frac{1}{6}$ th of what Nimal has, we get

$\frac{1}{4}x = \frac{1}{6}y$ as the second equation, which can be written as

$$\frac{1}{4}x - \frac{1}{6}y = 0 \text{ ——— } ②$$

When solving simultaneous equations involving fractions, it often helps to first clear the fractions and work with only integers. To clear the fractions we have to multiply each side of the equation by the least common multiple of the denominators.

Therefore, equation ① is multiplied by 6 which is the the least common multiple of the denominators 2, and 3 and equation ② is multiplied by 12 which is the least common multiple of the denominators 6 and 4.

$$① \times 6;\ 6 \times \frac{1}{2}x + 6 \times \frac{1}{3}y = 6 \times 20$$

$$\therefore\ 3x + 2y = 120 \text{ ——— } ③$$

$$② \times 12;\ 12 \times \frac{1}{4}x - 12 \times \frac{1}{6}y = 12 \times 0$$

$$3x - 2y = 0 \text{ ——— } ④$$

Now instead of ① and ② we can solve the two equivalent equations ③ and ④ involving only integers.

$$③ + ④ \quad (3x + 2y) + (3x - 2y) = 120 + 0$$

$$3x + 2y + 3x - 2y = 120$$

$$\frac{6x}{6} = \frac{120}{6}$$

$$x = 20$$

By substituting $x = 20$ in ④

$$3 \times 20 - 2y = 0$$

$$2y = 60$$

$$y = 30$$

$\therefore$ The amount Kamal has = Rs 20
The amount Nimal has = Rs 30

Note: In this problem, once we converted the fractions into integers, we solved for the unknown x by adding the equations. Alternatively we can make one unknown the subject of one equation and substitute in the other equation to obtain the solution. We will discuss one such example now.

Example 2

Solve.

$$\frac{1}{6}a - \frac{1}{5}b = -2 \text{ ——— } ①$$

$$\frac{1}{3}a + \frac{1}{4}b = 9 \text{ ——— } ②$$

The following steps will demonstrate how to solve simultaneous equations by substitution.

Let us make a the subject of equation ①.

$$\frac{1}{6}a - \frac{1}{5}b = -2$$

$$\frac{1}{6}a = -2 + \frac{1}{5}b$$

$$a = -12 + \frac{6}{5}b \quad \text{(multiplying both sides by 6)} \text{———} ③$$

We will substitute the value of a in equation ②.

$$\frac{1}{3}a + \frac{1}{4}b = 9$$

$$\frac{1}{3}(-12 + \frac{6}{5}b) + \frac{1}{4}b = 9$$

$$-4 + \frac{2}{5}b + \frac{1}{4}b = 9$$

We will simplify the fractions by taking 20 as the common denominator which is the least common multiple of 4 and 5.

$$\frac{8}{20}b + \frac{5}{20}b = 9 + 4$$

$$\frac{13}{20}b = 13$$

$$b = \frac{13 \times 20}{13}$$

$$b = 20$$

Substituting $b = 20$ in equation ③ (here you can substitute the value of b in either equation to find a), we get

$$a = -12 + \frac{6}{5}b$$

$$a = -12 + \frac{6}{5} \times 20$$

$$a = -12 + 24$$

$$a = 12$$

Therefore the solution to the problem is $a = 12$ and $b = 20$.

We can verify that the solution is correct by substituting $a = 12$ and $b = 20$ in the original equations.

Let us substitute $a = 12$ and $b = 20$ in the left side of equation ①.

$$\frac{1}{6}a-\frac{1}{5}b=-2$$

$$\text{Left hand side } =\frac{1}{6}a-\frac{1}{5}b$$
$$=\frac{1}{6}\times 12-\frac{1}{5}\times 20$$
$$=2-4$$
$$=-2$$
$$\therefore \text{ LHS} = \text{RHS}$$

The left hand side equals the right hand side of the equation after the substitution. That means $a = 12$ and $b = 20$ satisfy the equation $\frac{1}{6}a-\frac{1}{5}b=-2$

Similarly,
let us substitute $a = 12$ and $b = 20$ in the left side of equation ②.

$$\frac{1}{3}a+\frac{1}{4}b=9$$

$$\text{Left hand side } =\frac{1}{3}a+\frac{1}{4}b$$
$$=\frac{1}{3}\times 12+\frac{1}{4}\times 20$$
$$=4+5$$
$$=9$$
$$\therefore \text{ LHS} = \text{RHS}$$

The left hand side equals the right hand side of the equation after the substitution. That means $a = 12$ and $b = 20$ satisfy the equation $\frac{1}{3}a+\frac{1}{4}b=9$.
Therefore we can conclude that we have the correct answer.

Example 3

Solve :

$$\frac{1}{2}m+\frac{2}{3}n=1$$
$$\frac{5}{6}m+\frac{1}{3}n=4$$

Let us take:

$$\frac{1}{2}m+\frac{2}{3}n=1 \text{ ——— ①}$$
$$\frac{5}{6}m+\frac{1}{3}n=4 \text{ ——— ②}$$

As in example 1 we can either convert the fractions into integers or we can equate the fractional coefficients of a variable to solve the problem.

Let us equate the coefficients of the unknown n. We can do this by multiplying equation ② by 2

$$② \times 2 \quad \frac{10}{6}m + \frac{2}{3}n = 8 \text{ ——— } ③$$

$$③ - ① \quad \left(\frac{10}{6}m + \frac{2}{3}n\right) - \left(\frac{1}{2}m + \frac{2}{3}n\right) = 8 - 1$$

$$\frac{10}{6}m + \frac{2}{3}n - \frac{1}{2}m - \frac{2}{3}n = 7$$

$$\frac{10}{6}m - \frac{3}{6}m = 7$$

$$\frac{7}{6}m = 7$$

$$7m = 7 \times 6$$

$$m = 6$$

Let us substitute $m = 6$ in ①.

$$\frac{1}{2}m + \frac{2}{3}n = 1$$

$$\frac{1}{2} \times 6 + \frac{2}{3}n = 1$$

$$3 + \frac{2}{3}n = 1$$

$$\frac{2}{3}n = 1 - 3$$

$$\frac{2}{3}n = -2$$

$$2n = -6$$

$$n = -3$$

Therefore the answer is $m = 6$ and $n = -3$.

As in the previous problem, we can verify that the solution is correct by substituting the answers in the original equations.

Let us substitute $m = 6$ and $n = -3$ in the original equations.

$$\frac{1}{2}m + \frac{2}{3}n = 1 \text{ ——— ①}$$

$$\text{Left hand side } = \frac{1}{2}m + \frac{2}{3}n$$
$$= \frac{1}{2} \times 6 + \frac{2}{3} \times (-3)$$
$$= 3 - 2$$
$$= \underline{\underline{1}}$$

$\therefore$ Left hand side = Right hand side

$$\frac{5}{6}m + \frac{1}{3}n = 4 \text{ ——— ②}$$

$$\text{Left hand side } = \frac{5}{6}m + \frac{1}{3}n$$
$$= \frac{5}{6} \times 6 + \frac{1}{3} \times (-3)$$
$$= 5 - 1$$
$$= \underline{\underline{4}}$$

$\therefore$ Left hand side = Right hand side

Therefore we have the correct answer, $m = 6$ and $n = -3$.

Exercise13.1

1. Solve the following pairs of simultaneous equations.

(a) $\frac{3}{5}a + \frac{1}{3}b = 3$

$\frac{1}{2}a - \frac{1}{3}b = 8$

(b) $\frac{3}{5}x - \frac{1}{2}y = 9$

$\frac{1}{4}x - \frac{1}{2}y = 2$

(c) $\frac{1}{3}x + \frac{1}{2}y = 4$

$\frac{1}{2}x - y = 1$

(d) $\frac{2}{7}p - \frac{1}{3}q = 5$

$\frac{1}{2}p - 1\frac{2}{3}q = 12$

(e) $\frac{m}{4} + \frac{5n}{3} = 36$

$\frac{3m}{8} - \frac{5n}{12} = -2$

(f) $\frac{2x}{3} + \frac{3y}{2} = -1$

$4x - 5y = 22$

2. At a festival held in a school, the past pupils' association agreed to bear $\frac{1}{2}$ of the amount spent on refreshments and $\frac{1}{3}$ of the amount spent on decorations

Accordingly,the amount given by the past pupils' association was Rs 20 000. Rest of the amount spent on refreshments and decorations was given by the welfare society, which was Rs 30 000.

(i) Taking the amount spent on refreshment as x and the amount spent on decorations as y, construct a pair of simultanious equations to indicate the above information.

(ii) Solve the equations and find seperately the amount spent on refreshments and decorations.

13.2 Solving quadratic equations by factoring

You have learnt how to find the solutions (or roots) of a quadratic equation of the form $ax^2 + bx + c = 0$

Let us review a few a such examples.

Example1

Find the roots of the quadratic equation $x^2 - 5x + 6 = 0$.

$(x - 2)(x - 3) = 0$ (by factoring)

Then either $x - 2 = 0$ or $x - 3 = 0$.

Therefore $x = 2$ or $x = 3$

Therefore the roots of the equation are $x = 2$ and $x = 3$.

Example 2

Find the roots of $2x^2 + 3x - 9 = 0$

$$2x^2 + 6x - 3x - 9 = 0$$

$$2x(x + 3) - 3(x + 3) = 0$$

$$(2x - 3)(x + 3) = 0 \text{ (by factoring)}$$

$$2x - 3 = 0 \text{ or } x + 3 = 0$$

$$x = \frac{3}{2} \text{ or } x = -3$$

$\therefore$ $x = 1\frac{1}{2}$ and $x = -3$ are the roots of the equation.

Let us look at a more complex example.

Example 3

Find the roots of $\frac{3}{2x - 1} - \frac{2}{3x + 2} = 1$.

Here we cannot see the quadratic equation immediately. But when we clear the fractions, it can be set up as a quadratic equation. For this, we will consider the common denominator of the left hand side.(You can also do this by multiplying the whole equation by the least common multiple of $2x - 1$ and $3x + 2$)

$$\frac{3(3x + 2) - 2(2x - 1)}{(2x - 1)(3x + 2)} = 1 \text{ (writing the left hand side as a single fraction)}$$

$$3(3x + 2) - 2(2x - 1) = (2x - 1)(3x + 2) \text{ (cross multiplying)}$$
$$9x + 6 - 4x + 2 = 6x^2 + 4x - 3x - 2 \text{ (expanding the factors)}$$
$$6x^2 - 4x - 10 = 0 \text{ (simplifying)}$$
$$3x^2 - 2x - 5 = 0 \text{ (dividing the whole equation by 2)}$$
$$3x^2 - 5x + 3x - 5 = 0$$
$$x(3x - 5) + 1(3x - 5) = 0$$
$$(3x - 5)(x + 1) = 0$$
$$\therefore\ 3x - 5 = 0 \text{ or } x + 1 = 0$$
$$\therefore\ x = \frac{5}{3} \text{ or } x = -1$$
$$\therefore\ x = 1\frac{2}{3} \text{ or } x = -1$$
$$\therefore\ x = 1\frac{2}{3} \text{ and } x = -1 \text{ are the roots of this equation.}$$

Now let us consider a problem that can be solved by means of a quadratic equation.

Example 4

The product of two consecutive integers is 12. Find the two numbers.

Let us see how we can set up a quadratic equation to solve this problem.
Of the two consecutive integers, let us take x to be the smaller integer. Then the other integer is $x + 1$.

Therefore the consecutive integers can be written as x, $(x + 1)$.
Since the product of the two numbers is 12 we get,

$$x \times (x + 1) = 12.$$
$$\therefore\ x^2 + x - 12 = 0$$

After factoring we get

$$(x - 3)(x + 4) = 0.$$

This is, $x - 3 = 0$ or $x + 4 = 0$.
$\therefore$ $x = 3$ and $x = -4$ are the solutions of the above equation.

If we take $x = 3$ then the consecutive number is $(x + 1) = 3 + 1 = 4$.

If we take $x = -4$ the consecutive number is $(x + 1) = -4 + 1 = -3$.

Therefore there are two pairs of consecutive numbers whose product will be 12. They are 3, 4 and $-4, -3$.

We can verify the answer by substituting the two pairs of values in the quadratic equation $x^2 + x - 12 = 0$.

If we substitute $x = 3$ on the left hand side of $x^2 + x - 12 = 0$,

$$
\begin{aligned}
\text{LHS} &= x^2 + x - 12 \\
&= 3^2 + 3 - 12 \\
&= 9 + 3 - 12 \\
&= 12 - 12 \\
&= 0
\end{aligned}
$$

$\therefore$ LHS = RHS

If we substitute $x = -4$ on the left hand side,

$$
\begin{aligned}
\text{LHS} &= x^2 + x - 12 \\
&= (-4)^2 + (-4) - 12 \\
&= 16 - 4 - 12 \\
&= 16 - 16 \\
&= 0
\end{aligned}
$$

$\therefore$ LHS = RHS

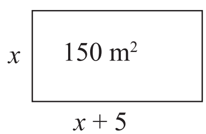
Therefore the solutions of the equation $x^2 + x - 12 = 0$ are indeed 3 and -4.

Example 5

The area of a rectangular plot of land is 150 square metres. The length of the plot is 5 metres more than the width. Let x be the width of the plot.

(i) Write the length of the plot in terms of x.
(ii) Set up a quadratic equation in x to represent the area of the plot x.
(iii) Solve the equation to find the length and the width of the plot.

(i) Length $= x + 5$.
(ii) The given data can be shown more clearly by aid of a drawing.



$$
\begin{aligned}
\text{Area} &= \text{length} \times \text{width} \\
&= (x + 5) \times x
\end{aligned}
$$

$$x(x + 5) = 150$$

This is the required equation.

(iii) Let us solve the above equation.

$$
\begin{aligned}
x(x + 5) &= 150 \\
x^2 + 5x - 150 &= 0 \\
(x - 10)(x + 15) &= 0
\end{aligned}
$$

$\therefore\ x - 10 = 0$ or $x + 15 = 0$

$\therefore\ x = +10$ or $x = -15$

The roots of the quadratic equation are $x = +10$ and $x = -15$.
But since a length cannot be negative, the only acceptable answer is $x = 10$.

Therefore the width of the rectangular plot = 10 m

The length of the rectangular plot = 15 m

We can verify the answer by substituting $x = 10$ in the quadratic equation $x^2 + 5x - 150 = 0$.

$$\begin{aligned} \text{LHS} &= x(x+5) \\ &= 10(10+5) \\ &= 10 \times 15 \\ &= 150 \end{aligned}$$

$\therefore$ LHS = RHS

Exercise 13.2

1. Solve the following quadratic equations.

(a) $x(x+5) = 0$ **(b)** $\frac{3}{4}x(x+1) = 0$ **(c)** $(x-4)(x+3) = 0$

(d) $x^2 - 2x = 0$ **(e)** $\frac{x^2}{2} = 3x$ **(f)** $x^2 + 7x + 12 = 0$

(g) $(x-2)(2x+3) = x^2 + 2x + 4$ **(h)** $\frac{4}{x} + \frac{3}{x+1} = 3$

(i) $\frac{2}{x-1} + \frac{3}{x+1} = 1$ **(j)** $x^2 - 4 = 0$

2. Solve each quadratic equation given below by factoring.

(Take $\sqrt{2} = 1.41$, $\sqrt{3} = 1.73$ and $\sqrt{5} = 2.23$)

(a) $x^2 - 12 = 0$ **(b)** $x^2 - 21 = 11$ **(c)** $x^2 + 17 = 37$

3. From the square of a certain number, if you subtract twice the number, the result is 15. Find the number.

4. The product of two consecutive even integers is 120. Find the two integers.

5. The length of a rectangular lamina is 3 cm more than the width. If the area is 88 cm^2, find the length and the width of the lamina.

6. A playing field measuring, 32 metres by 20 metres has a pathway outside the field all around it, of uniform width. The area of the pathway is 285 square metres.

   (i) Taking the width of the pathway to be x metres, set up a quadratic equation in x to represent the given information.

   (ii) Solve the quadratic equation to find the width of the pathway.

7. The hypotenuse of a right angled triangle is $2x + 1$ cm. The lengths of the other two sides are x cm and $x + 7$ cm respectively. Solve for x and find the lengths of all three sides of the triangle.

8. The sum of the first n terms of the arithmetic progression $-7, -5, -3, -1, \ldots$ is 105. Use your knowledge on progressions to answer the following.

   (i) Taking n to be the number of terms we are adding together, set up a quadratic equation in n for the sum of the first n terms.

   (ii) Solve the above equation to find the number of terms n you have added.

13.3 Solving a quadratic equation by completing the square

We have seen how to solve a quadratic equation by factorization. This method is useful only when the factorization is easy to do. Many quadratic expressions, such as $x^2 + 3x + 5 = 0$ and $2x^2 - 5x - 1 = 0$ cannot be factorized easily.To solve such quadratic equations we have to use alternative methods.One such method is to arrange the quadratic as a perfect square. This is known as solving the quadratic equation by completing the square.

Before we proceed, let us recall how we learnt to write an expression of the form $x^2 + bx$ as a perfect square.

Activity

Write the constant you should include to convert each of the following expressions into a perfect square and write it as a perfect square.

a. $x^2 + 6x + 9 \quad = (x + 3)^2$

b. $x^2 + 8x + \ldots \quad = \ldots\ldots\ldots\ldots$

c. $x^2 - 14x + \ldots = \ldots\ldots\ldots\ldots$

d. $x^2 + 3x + \ldots \quad = \ldots\ldots\ldots\ldots$

e. $(x + \ldots)^2 = x^2 + 8x + \ldots$

f. $(x + \ldots)^2 = x^2 + 2ax + \ldots$

g. $(x + b)^2 = x^2 + \ldots x + b^2$

h. $(x + m)^2 = x^2 + \ldots x + m^2$

First we will complete the square of a quadratic equation which can also be solved by factoring.

Example 1

Solve $x^2 + 2x - 3 = 0$ by completing the square.

First we will keep all terms containing x on one side and move the constant to the right. Then we get

$$x^2 + 2x - 3 = 0$$
$$x^2 + 2x = 3$$

To write the left hand side as a perfect square, take half of the x-term coefficient and square it and add this value to both sides. That is, in this problem we have to add +1 to both sides.

$$x^2 + 2x + 1 = 3 + 1$$
$$(x + 1)^2 = 4$$

Take the square root of both sides. We have to allow for both positive and negative values.

$$x + 1 = \pm\sqrt{4}$$
$$x + 1 = \pm 2$$
$$x = \pm 2 - 1$$

Then $x = +2 - 1$ or $x = -2 - 1$

$x = 1$ or $x = -3$

Therefore the solutions to the above equation are $x = 1$ and $x = -3$.

Now let us look at another example.

Example 2

Solve $x^2 - 4x + 1 = 0$ by completing the square.

$$x^2 - 4x + 1 = 0$$
$$x^2 - 4x - 1$$
$$x^2 - 4x + 4 = -1 + 4$$
$$(x - 2)^2 = 3$$
$$x - 2 = \pm\sqrt{3} \quad \text{(taking the square root of both sides)}$$
$$x = 2 \pm\sqrt{3}$$
$$x = 2 + \sqrt{3} \quad \text{or} \quad x = 2 - \sqrt{3}.$$

Suppose it is given that $\sqrt{3}$ is approximately 1.73

Then $x = 2 + 1.73$ or $x = 2 - 1.73$

$x = 3.73$ or $x = 0.27$

Then the solutions of this equation are $x = 3.73$ and $x = 0.27$.

Example 3

Find the roots of $2x^2 + 6x - 5 = 0$ by completing the square.

Here it is easy if we first divide all the terms by 2 to make the leading coefficient (coefficient of the x^2 term) one.

$$2x^2 + 6x - 5 = 0$$
$$x^2 + 3x - \frac{5}{2} = 0 \quad \text{(now the leading coefficient is one)}$$
$$x^2 + 3x = \frac{5}{2}$$
$$x^2 + 3x + \left(\frac{3}{2}\right)^2 = \frac{5}{2} + \left(\frac{3}{2}\right)^2$$
$$\left(x + \frac{3}{2}\right)^2 = \frac{5}{2} + \frac{9}{4}$$
$$\left(x + \frac{3}{2}\right)^2 = \frac{+10 + 9}{4}$$
$$\left(x + \frac{3}{2}\right)^2 = \frac{+19}{4}$$
$$x + \frac{3}{2} = \frac{\pm\sqrt{19}}{2}$$
$$x = \frac{+\sqrt{19} - 3}{2} \text{ or } x = \frac{-\sqrt{19} - 3}{2}$$

Suppose 4.36 is given as an approximate value of $\sqrt{19}$.

Then $\quad x = \dfrac{4.36 - 3}{2}$ or $x = \dfrac{-4.36 - 3}{2}$

$$x = 0.68 \text{ or } x = -3.68$$

The roots of the equation are $x = 0.68$ and $x = -3.68$.

Exercise13.3

1. Solve the following quadratic equations by completing the square.

(Take $\sqrt{2} = 1.41$, $\sqrt{3} = 1.73$, $\sqrt{5} = 2.23$, $\sqrt{6} = 2.44$, $\sqrt{13} = 3.6$, $\sqrt{17} = 4.12$ and $\sqrt{57} = 7.54$)

(a) $x^2 - 2x - 4 = 0$ **(b)** $x^2 + 8x - 2 = 0$ **(c)** $x^2 - 6x = 4$

(d) $x^2 + 4x - 8 = 0$	**(e)** $x(x + 8) = 8$	**(f)** $x^2 + x = 4$
(g) $2x^2 + 5x = 4$	**(h)** $3x^2 = 3x + \frac{1}{2}$	**(i)** $\frac{2}{x+3} + \frac{1}{2x+3} = 1$

13.4 Solving quadratic equations by using the quadratic formula

An easy method of solving a quadratic equation of the form $ax^2 + bx + c = 0$ is by using the quadratic formula.

The quadratic formula is derived by completing the square on a general quadratic equation $ax^2 + bx + c = 0$.

Once the quadratic formula is derived, it is no longer necessary to use the process of completing the square to solve "each" quadratic equation.

$$ax^2 + bx + c = 0$$

$$ax^2 + bx = -c$$

$$\frac{ax^2}{a} + \frac{bx}{a} = -\frac{c}{a} \quad (\text{ dividing by } a)$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \quad \left(\text{squaring } \frac{1}{2} \text{ of } \frac{b}{a} \text{ and adding to both sides}\right)$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

(writing the left hand side as a perfect square and arranging the terms on the right hand side)

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

(simplifying the right hand side by using a common denominator)

Then $x + \dfrac{b}{2a} = \pm\sqrt{\dfrac{b^2 - 4ac}{4a^2}}$

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(simplifying by using a common denominator)

Therefore
to solve a quadratic equation of the form $ax^2 + bx + c = 0$

we can use the quadratic formula $\boxed{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$

We get two values (roots) for x due to the positive and negative signs.

Here a is the coefficient of x^2

b is the coefficient of x and

c is the constant term.

Example 1

Solve $2x^2 + 7x + 3 = 0$ by using the quadratic formula.

To solve the equation $2x^2 + 7x + 3 = 0$, we can take $a = 2$, $b = 7$ and $c = 3$ and substitute these values in the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-7 \pm \sqrt{7^2 - 4 \times 2 \times 3}}{2 \times 2}$$

$$= \frac{-7 \pm \sqrt{49 - 24}}{4}$$

$$= \frac{-7 \pm \sqrt{25}}{4}$$

$$= \frac{-7 \pm 5}{4}$$

$$x = \frac{-7 + 5}{4} \quad \text{or } x = \frac{-7 - 5}{4}$$

$$x = -\frac{1}{2} \text{ or } x = -3$$

$x = -\dfrac{1}{2}$ and $x = -3$ are the solutions of the above equation.

Example 2

Solve $4x^2 - 7x + 2 = 0$ by using the quadratic formula, and find its roots

Take $\sqrt{17} = 4.12$.

$$4x^2 - 7x + 2 = 0$$

Substitute $a = 4$, $b = -7$, $c = 2$. (According to the equation $ax^2 + bx + c = 0$)

in $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

$$= \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \times 4 \times 2}}{2 \times 4}$$

$$= \frac{7 \pm \sqrt{49 - 32}}{8}$$

$$= \frac{7 \pm \sqrt{17}}{8}$$

$$= \frac{7 \pm 4.12}{8} \quad (\sqrt{17} = 4.12 \text{ is given})$$

$$x = \frac{7 + 4.12}{8} \quad \text{or} \quad x = \frac{7 - 4.12}{8}$$

$$x = \frac{11.12}{8} \quad \text{or} \quad x = \frac{2.88}{8}$$

$x = 1.39$ or $x = 0.36$

$x = 1.39$ and $x = 0.36$ are the roots of the equation.

Example 3

Solve $x^2 + 2x - 1 = 0$ by using the quadratic formula and find the roots accurate to the second decimal place.

(Take $\sqrt{2} = 1.414$).

$a = 1, b = 2, c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$

$$= \frac{-2 \pm \sqrt{4 + 4}}{2}$$

$$= \frac{-2 \pm \sqrt{8}}{2}$$

$$= \frac{-2 \pm \sqrt{4 \times 2}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2}$$

$$= \frac{-2 \pm 2 \times 1.414}{2}$$

$$= \frac{-2 \pm 2.828}{2}$$

$$x = \frac{-2 + 2.828}{2} \quad \text{or} \quad x = \frac{-2 - 2.828}{2}$$

$$= \frac{0.828}{2} \qquad x = \frac{-4.828}{2}$$

$$x = 0.414 \text{ or } x = -2.414$$

The roots are $x = 0.41$ and $x = -2.41$.

Exercise13.4

1. Solve the following quadratic equations using the quadratic formula and give your answers accurate to the first decimal place.

(Take $\sqrt{3} = 1.73$, $\sqrt{17} = 4.12$ and $\sqrt{29} = 5.38$)

(a) $x^2 - 6x - 3 = 0$ **(b)** $x^2 - 7x + 5 = 0$ **(c)** $2x^2 - x - 2 = 0$

(d) $2x^2 - 5x + 1 = 0$ **(e)** $3x^2 - 4x - 7 = 0$

Miscellaneous Exercise

1. When three times a number which is postive is subtracted from the square of that number the answer is 28. Find the number.

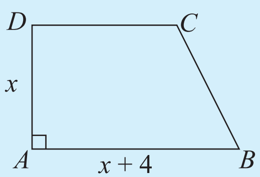
2. The product of two consecutive odd integers is 99. Find the two integers.

3. The area of a rectangular sheet is 44 square centimetres. The length of the sheet is 6 centimetres more than the width. Let x be the width of the sheet in centimetres.

(i) Set up a quadratic equation in x to represent the given information.

(ii) Solve the equation to find the value of x accurate to the first decimal place. (Take $\sqrt{53} = 7.28$). Hence find the length of the sheet.

4.



$ABCD$ is a trapezium with $AD = CD$.

(i) If the area of the trapezium is 12 cm^2 show that x satisfies $x^2 + 2x - 12 = 0$.

(ii) Solve the quadratic equation in (i) above by completing the square or by some other method and find the value of x to the nearest first decimal place.

5. The squares of three consecutive natural numbers add up to 149.Taking x to be the number in the middle, find the largest number.

6. In a right angled triangle the lengths of the two sides containing the right angle are $5x$ cm and $(3x - 1)$ cm. If the area is given as 60 square centimetres, set up a quadratic equation in x and solve for x. Thereby find the length of each side of the triangle.

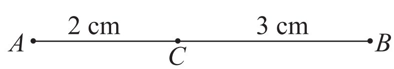
7. A man bought a certain number of mangoes for Rs 600.If the price of a mango was one rupee less, then he could have bought 20 more mangoes. Find the number of mangoes he bought.

14 Equiangular Triangles

By studying this lesson you will be able to

- understand the meaning of equiangular and similar figures,
- identify the theorem "A line drawn parallel to a side of a triangle, divides the other two sides proportionally",
- identify the converse theorem "If a straight line divides two sides of a triangle proportionally then that line is parallel to the remaining side",
- identify the theorem "Corresponding sides of equiangular triangles are proportional",
- identify the converse theorem "If the corresponding sides of two triangles are proportional then those triangles are equiangular".

Ratios of lengths



The figure illustrates the line segment AB. The point C lies on AB such that $AC = 2$ cm and $CB = 3$ cm. Point C divides AB into two line segments AC and CB. Then, the ratio of AC to CB can be written using their lengths as follows.

$AC : CB = 2 : 3$

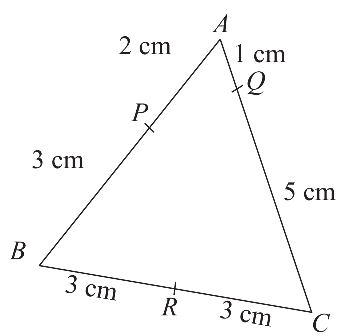
Similarly $AC : AB = 2 : 5$ (since $AB = 5$cm)

hence $CB : AC = 3 : 2$

$CB : AB = 3 : 5$.

The ratio of the lengths of line segments should be written in the order of the line segments relevant to that ratio.

Consider the triangle ABC in the following figure.



As shown in the figure, the points P, Q and R lie on each side of the triangle ABC. Then the ratios can be written as follows.

(i) $AP : PB = 2 : 3$, $AP : AB = 2 : 5$, $PB : AP = 3 : 2$
(ii) $AQ : QC = 1 : 5$, $AQ : AC = 1 : 6$, $QC : AQ = 5 : 1$
(iii) $BR : RC = 3 : 3 = 1 : 1$, $BR : BC = 3 : 6 = 1 : 2$

We have already learnt that ratios can be written as fractions.

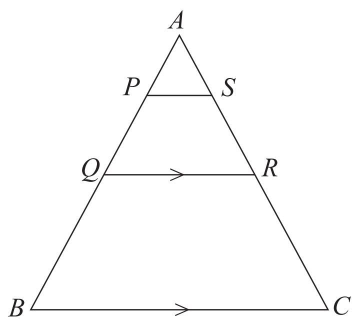
Therefore $AQ : QC = 1 : 5$ can be also written as $\frac{AQ}{QC} = \frac{1}{5} = 0.2$

14.1 Dividing two sides of a triangle by a line drawn parallel to the other side

Let us do the following activity to find out about the ratios in which two sides of a triangle are divided by a line drawn parallel to the other side.

Activity

- Draw a triangle ABC with AB = 6cm and choosing any length for the other two sides.
- Mark the points P and Q on AB such that AP = 2 cm and AQ = 3 cm.
- Using the set square or using any other method, draw a line through Q parallel to BC and name the point that it meets the line AC as R.



- Measure AR and RC.
- Similarly draw another line parallel to BC through P and mark the point that it meets the line AC as S.
- Measure AS and SC.
- Now complete the following table.

State	Ratio between segments of AB	Ratio between segments of AC	Relationship between the two ratios
Parallel line through Q	$\frac{AQ}{QB} = \frac{3}{3} = 1$	$\frac{AR}{RC} =$	
Parallel line through P	$\frac{AP}{PB} = \frac{2}{4} = 0.5$	$\frac{AS}{SC} =$	

- Similarly check the relationship between the ratios in which the two sides of a right angled triangle and obtuse angled triangle are divided by a line drawn parallel to the other side.

Check whether your results agree with the following sentence.

A line drawn parallel to one side of a triangle divides the other two sides in equal ratios.

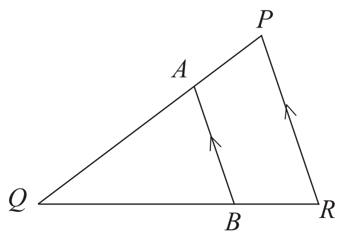
The above result can be given as a theorem in geometry.

Theorem:

A line drawn parallel to a side of a triangle divides the other two sides proportionally.

As an example, the line AB drawn parallel to the side PR of the triangle PQR is shown in the figure.
Then according to the theorem,

(i) $QA : AP = QB : BR$ hence, $\frac{QA}{AP} = \frac{QB}{BR}$.



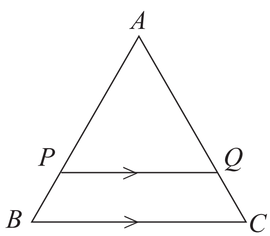


Figure (i)

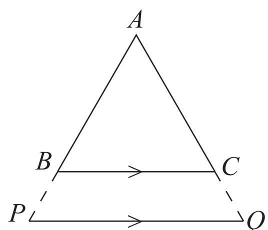


Figure (ii)

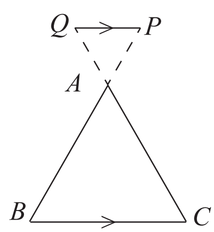


Figure (iii)

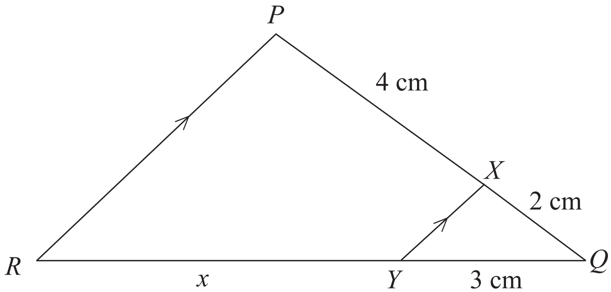
As shown in figure (i), the line PQ drawn parallel to BC internally divides the sides AB and AC. But in figure (ii) and figure (iii), the line PQ, drawn parallel to BC meets other two sides produced at P and Q. In these kinds of situations, it is said that PQ externally intersects AB and BC. Irrespective of whether the sides are divided internally or externally, the theorem is valid.

Hence for all three figures above, $\dfrac{AP}{PB} = \dfrac{AQ}{QC}$.

Now consider the following examples with calculations using the theorem.

Example 1

In triangle PQR, the line XY is drawn parallel to PR. Find the length of RY if $PX = 4$ cm, $XQ = 2$ cm, and $YQ = 3$ cm.



Let x be the length of RY.
Then, since XY is drawn parallel to PR,

according to the theorem $\dfrac{RY}{YQ} = \dfrac{PX}{XQ}$

hence $\dfrac{x}{3} = \dfrac{4}{2}$

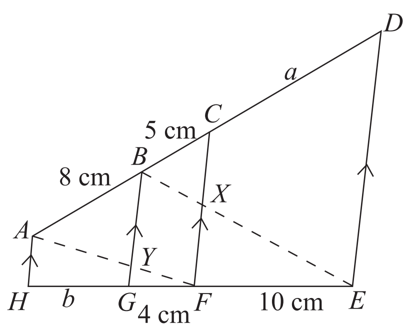
$\therefore \quad 2x = 4 \times 3$

$\therefore \quad x = 6$

$\therefore$ The length of RY is 6 cm.

Example 2

Based on the information given in the figure, find the values of a and b.



Let us join BE first.

In triangle BED, as $DE//CX$, according to the theorem, CX divides the sides BD and BE proportionally.

Hence, $\dfrac{BC}{CD} = \dfrac{BX}{XE}$

Hence, $\dfrac{5}{a} = \dfrac{BX}{XE}$ ——①

Now, in triangle BGE, as $BG//XF$, according to the theorem, XF divides the sides EB and EG proportionally.

Hence,

$$\frac{BX}{XE} = \frac{GF}{FE}$$

$$\therefore \quad \frac{BX}{XE} = \frac{4}{10} \quad \text{——②}$$

From ① and ②

$$\frac{5}{a} = \frac{4}{10}$$

Hence, $4a = 50$

$$\therefore \quad a = \frac{50}{4}$$

$$= 12.5 \text{ cm}$$

Similarly, let us join AF.

In triangle ACF, $\dfrac{AB}{BC} = \dfrac{AY}{YF}$

$$\frac{8}{5} = \frac{AY}{YF} \quad \text{——③}$$

In triangle AHF, $\dfrac{AY}{YF} = \dfrac{HG}{GF}$

$$\frac{AY}{YF} = \frac{b}{4} \quad \text{——④}$$

From ③ and ④,

$$\frac{b}{4} = \frac{8}{5}$$

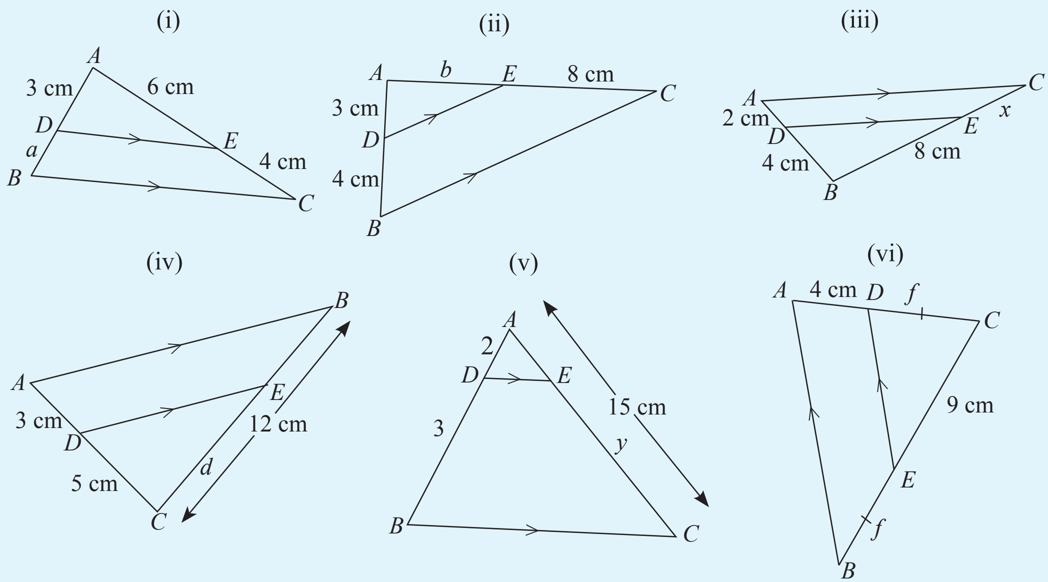
hence $5b = 32$

$$\therefore \quad b = \frac{32}{5}$$
$$= 6.4 \text{ cm}$$

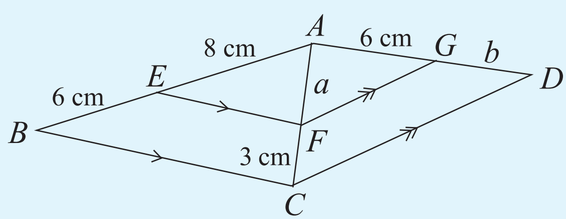
Now establish what was learnt by doing the calculations in the following exercise.

Exercise14.1

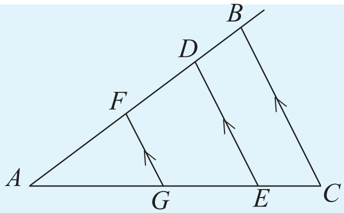
1. In each of the following figures, the lengths of some line segments are denoted by unknown terms. Find the values of those unknown terms.



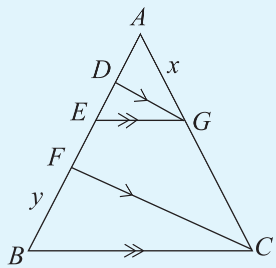
2. Based on the information in the figure, find the values of a and b.



3. In the given figure, $FG//DE//BC$. Also, $AF = 6$ cm, $DB = 3$ cm, $AG = 8$ cm and $GE = 8$ cm. Find the lengths of the line segments FD and EC.



4. In the given figure, $DG//FC$ and $EG//BC$. $AD = 6$ cm, $DE = 4$ cm, $EF = 5$ cm and $GC = 18$ cm. Find the values of x and y.

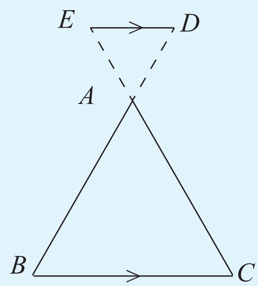


5. As shown in the figure, in triangle ABC, the line ED drawn parallel to BC externally divides the sides BA and CA produced. Also $AE = 2$ cm, $AD = 3$ cm and $AC = 4$ cm. The length of the line segment AB is denoted by x.

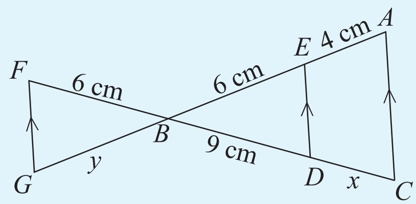
(i) Fill in the blanks.

DB : .... = ... : EA

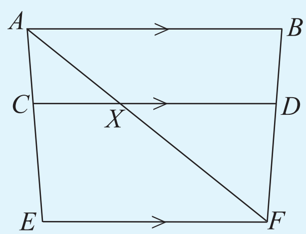
(ii) Find the value of x.



6. Based on the given information in the figure, find the values of x and y.



7. In the given figure $AB//CD//EF$. $AC = 3$ cm, $CE = 5$ cm and $BF = 12$ cm. Find the lengths of BD and DF.



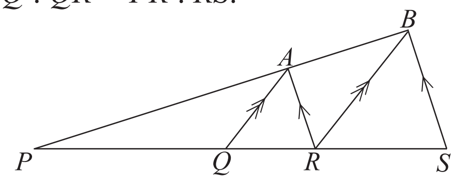
8. In a triangle ABC, the bisector of $B\hat{C}A$ meets AB at X. The point P lies on BC such that $PX = PC$. If $PX = 9$ cm, $BX = 5$ cm and $AX = 6$ cm, then find the length of BC.

14.2 More on dividing two sides of a triangle proportionally

In this section let us consider how riders are proved using the theorem "A line drawn parallel to a side of a triangle divides the other two sides proportionally"

Example 1

In the given figure, *PQRS* and *PAB* are straight lines. *BS*//*AR* and *BR*//*AQ*. Prove that $PQ : QR = PR : RS$.



Proof: In the triangle *PBR*, as *AQ* is parallel to *BR*, according to the theorem,

$PA : AB = PQ : QR$ ——— ①

In the triangle *PBS* as *AR* is parallel to *BS*, according to the theorem

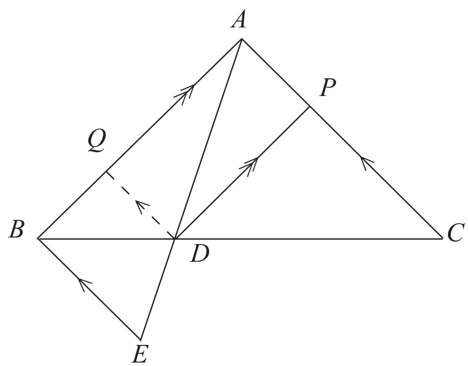
$PA : AB = PR : RS$ ——— ②

From ① and ②

$PQ : QR = PR : RS$

Example 2

The point *D* lies on the side *BC* of the triangle *ABC*. The line *BE* drawn parallel to *AC* meets *AD* produced at *E*. The line drawn from *D*, parallel to *AB* meets *AC* at *P*. Prove that $CP : AP = AD : DE$.



Here, as in the above example, choose two triangles and in each triangle draw a line parallel to a relevant side of the triangle. We choose triangles *ABE* and *ABC*, because there is a common side to these two triangles.

But there is no line parallel to a side of the *ABC* triangle. Therefore let us construct such a line.

Construction: Draw line DQ parallel to BE such that it meets AB at Q. (Then AC, QD, and BE are parallel to each other)

Proof :

In the triangle ABC, PD is parallel to AB. Therefore by the theorem,

$$CP : PA = CD : DB \text{ ——— } ①$$

In the triangle ABC, QD is parallel to AC. Therefore by the theorem,

$$AQ : QB = CD : DB \text{ ——— } ②$$

In the triangle ABE, QD is parallel to BE. Therefore by the theorem,

$$AQ : QB = AD : DE \text{ ——— } ③$$

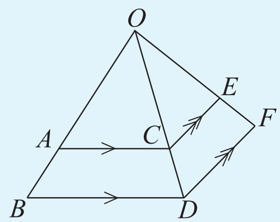
From the equations ①, ② and ③

$$CP : PA = CD : DB = AQ : QB = AD : DE.$$

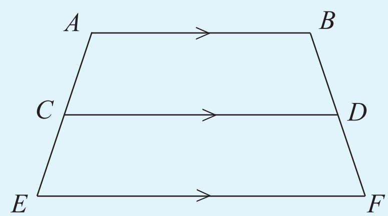
$$\therefore \quad CP : PA = AD : DE$$

Exercise 14.2

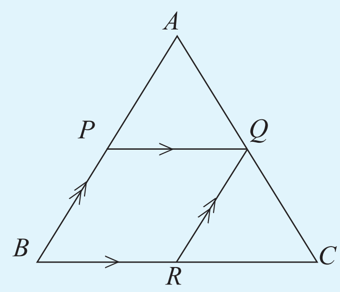
1. Based on the information in the figure, show that $OA : AB = OE : EF$.



2. Based on the given information in the figure, prove that $AC : CE = BD : DF$.

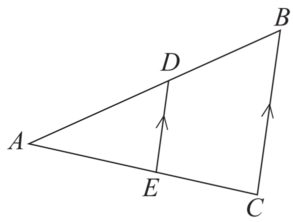


3. Based on the given information in the figure, prove that $AP : AB = BR : BC$.



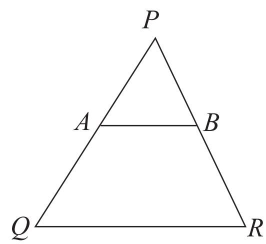
4. In triangle PQR, the point A lies on QR. A line drawn through A, parallel to PR, meets PQ at B. The line RCD drawn from R, intersects AB at C and PQ at D. If $D\hat{B}C = B\hat{C}D$ then prove that $\dfrac{QA}{AR} = \dfrac{QB}{CR}$.

14.3 The converse of the theorem “a line drawn parallel to a side of a triangle divides the other two sides proportionally ”



The above theorem says that, in triangle ABC, a line drawn parallel to BC divides AB and AC in equal ratios.

That is, since $BC//DE$, $AD : DB = AE : EC$. Let us understand the converse of this theorem by considering the triangle PQR shown in the figure.



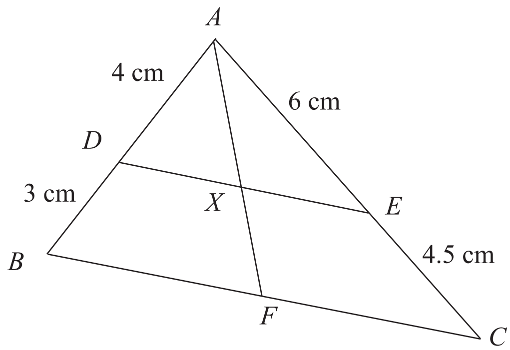
Here the line AB intersects the sides PQ and PR. The ratios of the line segments of each side are $PA : AQ$ and $PB : BR$.

If these two ratios are equal, that is, if $PA : AQ = PB : BR$ then the line AB, which intersects those two sides at P and Q, is parallel to the other side QR. This is the converse of the theorem we have already learnt in this lesson. This result can be stated as a theorem.

The converse of the above theorem:
If a line divides two sides of a triangle proportionally, then that line is parallel to the other side.

Given below are some examples with calculations and proved riders related to the theorem.

Example 1



Based on the information in the figure, find the value of $AX : XF$.

Consider the triangle ABC. Then, $AD : DB = 4 : 3$ and

$$AE : EC = 6 : 4.5 = 4 : 3.$$

$$\therefore \quad AD : DB = AE : EC$$

$\therefore$ The line DE divides AB and AC proportionally.

$\therefore$ By the converse of the theorem, $DE//BC$.

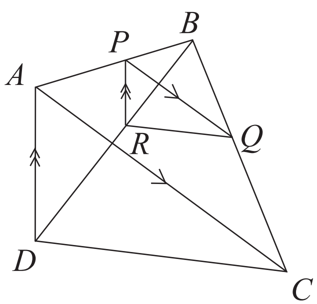
Therefore in triangle ABF, since $DX//BF$,

$AD : DB = AX : XF$

But $AD : DB = 4 : 3$

Hence $AX : XF = \underline{\underline{4 : 3}}$

Example 2



The point P lies on the side AB of the quadrilateral $ABCD$. The line drawn through P, parallel to AC, meets BC at Q. Similarly, the line drawn through P, parallel to AD, meets BD at R. Prove that $RQ//DC$.

Proof :

In triangle ABD, since PR is parallel to the side AD,

$BP : PA = BR : RD$ ——— ①

In triangle ABC, since PQ is parallel to the side AC,

$BP : PA = BQ : QC$ ——— ②

From equations ① and ②

$BR : RD = BQ : QC$.

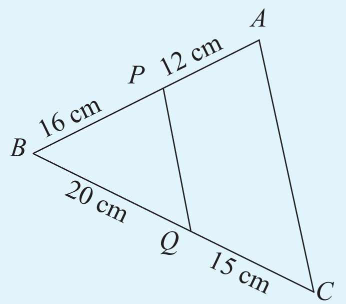
$\therefore$ In triangle BDC, the line RQ divides the sides BD and BC proportionally.

$\therefore$ $RQ//DC$ (by the converse theorem)

For the following exercises use the converse theorem given above.

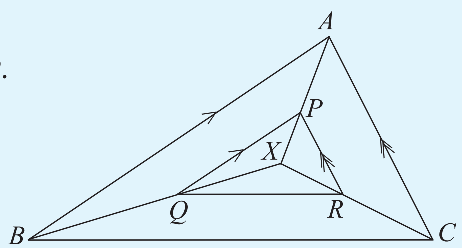
Exercise 14.3

1. Based on the information given in the figure, show that AC is parallel to PQ.

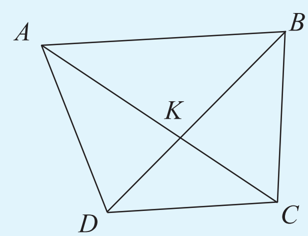


2. In triangle ABC, the point P lies on the side AB and the point Q lies on the side AC such that $AP : PB = AQ : QC$. Prove that $Q\hat{P}B + P\hat{B}C = 180^\circ$.

3. In the given figure, $AC//PR$ and $AB//PQ$. Prove that $BC//QR$.

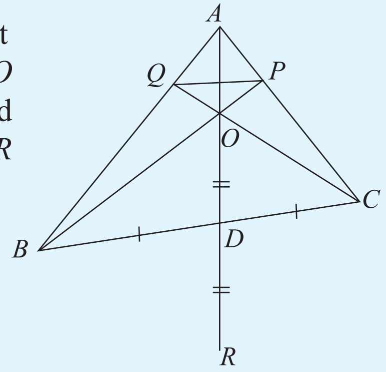


4. In the quadrilateral $ABCD$ given in the figure, the diagonals AC and BD intersect at K. If $AK = 4.8$ cm, $KC = 3.2$ cm, $BK = 3$ cm and $KD = 2$ cm, then show that DC is parallel to AB.(Hint: In triangle KDC, take that the points A and B lie on DK and CK produced respectively.)



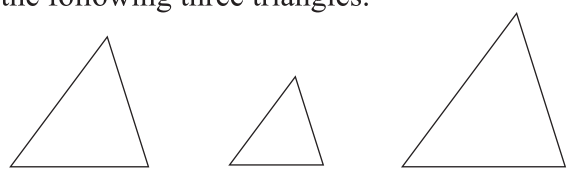
5. In the triangle ABC given in the figure, the midpoint of the side BC is D. The point O lies on AD. BO produced intersects AC at P and CO produced intersects AB at Q. The line AD is produced to R such that $OD = DR$. Prove that,

(i) $BRCO$ is a parallelogram.
(ii) $AQ : QB = AO : OR$.
(iii) $QP//BC$.

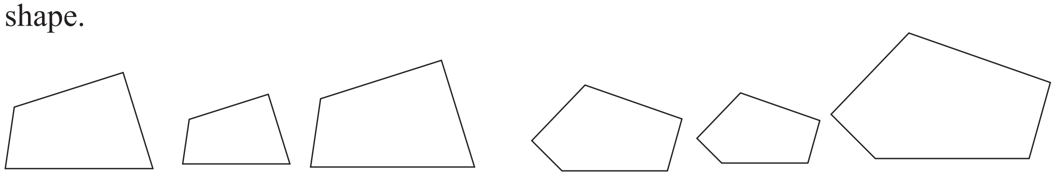


14.4 Similar figures and equiangular figures

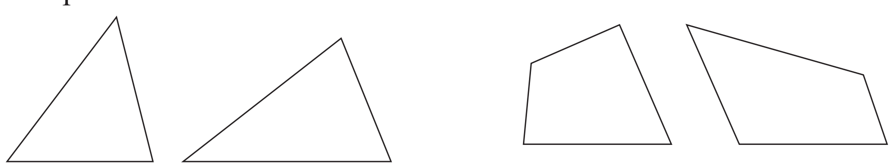
Carefully observe the following three triangles.



In day to day language we say that these three triangles are of the "same shape". The following figure illustrates three quadrilaterals and three pentagons of the same shape.



But the following pair of triangles as well as pair of quadrilaterals are not of the same shape.

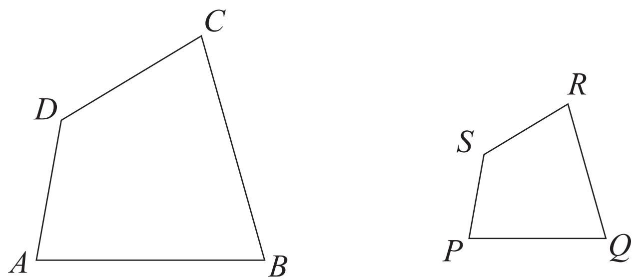


Did you think of what is meant by "shape" here? In mathematics things should be defined precisely. It is necessary to give a precise definition for "shape". "Similar figure" is the phrase in mathematics equivalent to "same shape" in day to day language. Here we consider only the similarity of polygons.

Two polygons are said to be similar if;

1. the angles of one polygon are equal to the angles of the other and,
2. the corresponding sides of the polygons are proportional.

As an example, consider the following two quadrilaterals *ABCD* and *PQRS.*



In these two quadrilaterals,

if $\hat{A} = \hat{P}, \hat{B} = \hat{Q}, \hat{C} = \hat{R}, \hat{D} = \hat{S}$ and

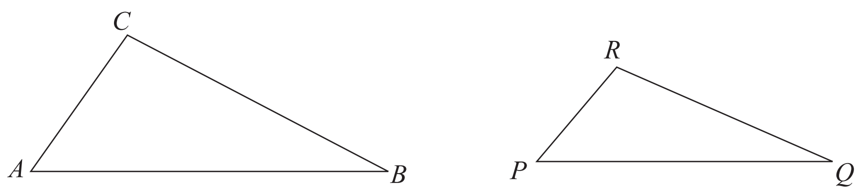
$$\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CD}{RS} = \frac{DA}{SP},$$

then the quadrilaterals *ABCD* and *PQRS* are similar.

In this lesson we hope to study more about similar triangles.
In the two triangles *ABC* and *PQR* given below,

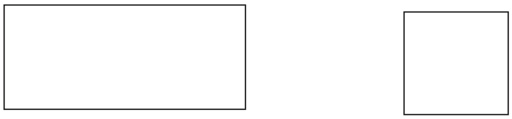
if $\hat{A} = \hat{P}, \hat{B} = \hat{Q}, \hat{C} = \hat{R}$

and $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$, then by definition the two triangles are similar.



There is an important result related to the similarity of triangles. That is, if the angles of one triangle are equal to the angles of another triangle, then those two triangles are similar. In otherwords, if three angles of a triangle are equal to the three angles of another triangle then the corresponding sides of those two triangles are proportional. Therefore to find out whether two triangles are similar, it is sufficient to check only the angles of the two triangles. As an example, in the above two triangles, if $\hat{A} = \hat{P}, \hat{B} = \hat{Q}$ and $\hat{C} = \hat{R}$, then $\frac{AB}{PQ} = \frac{BC}{QR} = \frac{CA}{RP}$.

This result is not true for polygons which are not triangles. For example, in the following two quadrilaterals, the angles of one quadrilateral are equal to the angles of the other quadrilateral. Each of these angles is 90°. However one of the quadrilaterals is a rectangle and the other is a square. Therefore their sides are not proportional and hence the two quadrilaterals are not similar.



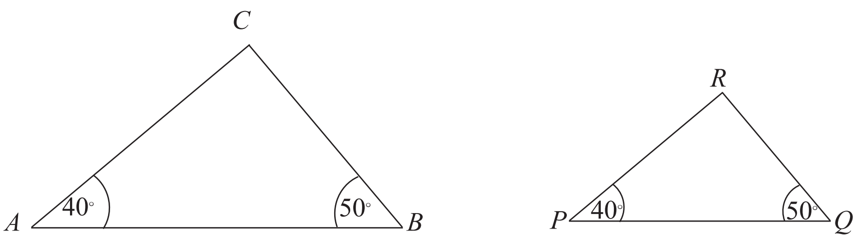
If the angles of two polygons are equal, then those polygons are said to be equiangular. According to the above discussion, two equiangular triangles are similar too. Let us use this result as a theorem without proof.

Theorem on equiangular triangles:
If two triangles are equiangular then the corresponding sides are proportional.

Do the following activity to understand more about this result.

Activity

- Using the protractor, draw two different sized triangles, each with interior angles 40°, 50° and 90°. As shown below, name the triangles as ABC and PQR.



- Find the ratios(as fractions) between the corresponding sides of the two triangles. That is, find the values of $\frac{AB}{PQ}$, $\frac{BC}{QR}$ and $\frac{CA}{RP}$ separately.
- Check whether these three values are equal.(You can have minor errors because of errors in the measurements)

According to the above activity, you can understand that if two triangles are equiangular then they are similar.

Note:

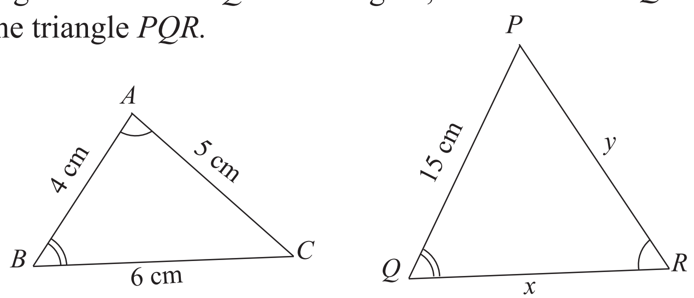
1.If restricted to triangles, similar and equi-angular mean the same thing.

2.It is clear that two congruent triangles are similar. But two similar triangles may not be congruent.

3. If two angles of one triangle are equal to two angles of another triangle, then the remaining pair of angles are also equal to each other. The reason for this is because the sum of the three angles of a triangle is equal to 180°. Therefore, if two angles of one triangle are equal to two angles of another triangle, then they are equiangular.

Example 1

In the two triangles *ABC* and *PQR* in the figure, $\hat{A} = \hat{R}$ and $\hat{B} = \hat{Q}$. Find the values of *x* and *y* in the triangle *PQR*.



In the two triangles *ABC* and *PQR*

$\hat{A} = \hat{R}$ and $\hat{B} = \hat{Q}$

$\therefore\ \hat{C} = \hat{P}$ (since the sum of the three interior angles of a triangle is 180°)

$\therefore$ The triangles *ABC* and *PQR* are equiangular.

$\therefore$ Corresponding sides are propotional.

Hence; $\dfrac{BC}{PQ} = \dfrac{AB}{QR}$

$$\therefore \quad \frac{6}{15} = \frac{4}{x}$$

$6x = 15 \times 4$ (By cross multiplication)

$$\therefore \quad x = \frac{15 \times 4}{6}$$

$$= \underline{\underline{10 \text{ cm}}}$$

$$\frac{BC}{PQ} = \frac{AC}{PR}$$

$$\frac{6}{15} = \frac{5}{y}$$

$$6y = 15 \times 5$$

$$y = \frac{15 \times 5}{6}$$

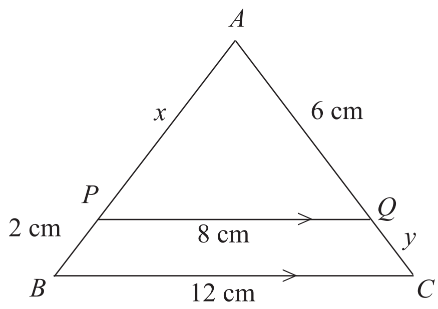
$$= \underline{\underline{12.5 \text{ cm}}}$$

Example 2

In the triangle ABC, the line PQ is drawn parallel to the side BC.

(i) Show that the triangles ABC and APQ are equiangular.

(ii) Find the values of x and y.



(i) In triangles ABC and APQ,

$A\hat{B}C = A\hat{P}Q$ (corresponding angles, $BC//PQ$)

$A\hat{C}B = A\hat{Q}P$ (corresponding angles, $BC//PQ$)

$\hat{A}$ is common to both triangles

$\therefore$ The triangles ABC and APQ are equiangular.

(ii) Since ABC and APQ are equiangular triangles, according to the theorem, the corresponding sides are proportional.

$$\therefore \frac{BC}{PQ} = \frac{AB}{AP}$$

$$\therefore \frac{12}{8} = \frac{x+2}{x}$$

$$12x = 8(x+2)$$

$$12x = 8x + 16$$

$$12x - 8x = 16$$

$$4x = 16$$

$$\underline{\underline{x = 4 \text{ cm}}}$$

$$\therefore \frac{BC}{PQ} = \frac{AC}{AQ}$$

$$\therefore \frac{12}{8} = \frac{6+y}{6}$$

$$8(6+y) = 6 \times 12$$

$$48 + 8y = 72$$

$$8y = 72 - 48$$

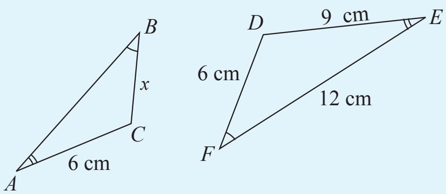
$$8y = 24$$

$$\underline{\underline{y = 3 \text{ cm}}}$$

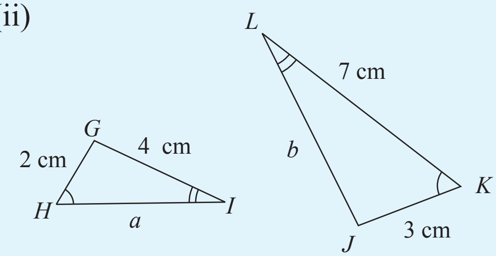
Exercise14.4

1. For each pair of triangles given below, find the lengths of the sides represented by unknowns.

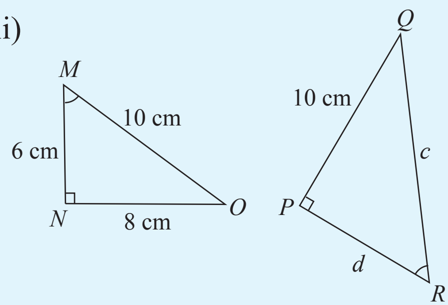
(i)



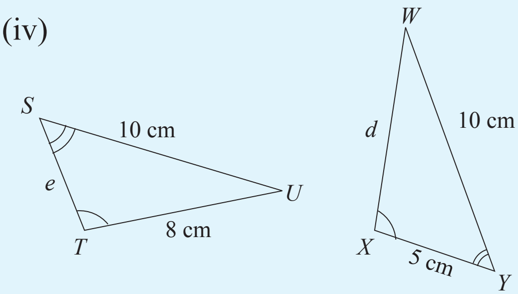
(ii)



(iii)

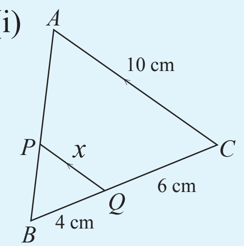


(iv)

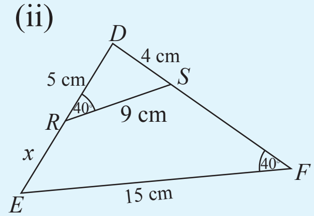


2. Show that each pair of triangles in each of the following figures is equiangular and find the lengths of sides represented by unknowns.

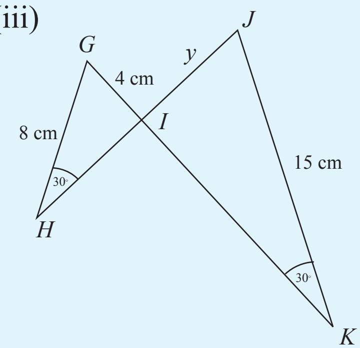
(i)



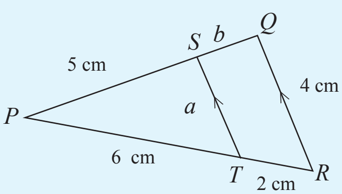
(ii)



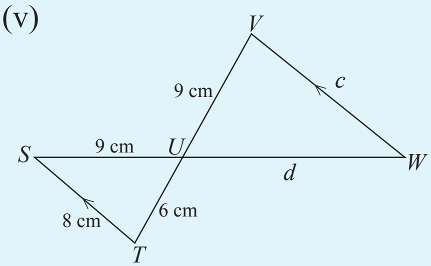
(iii)



(iv)



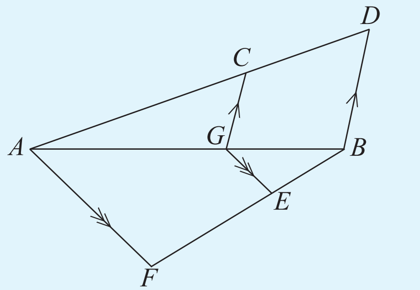
(v)



3. Based on the information given in the figure

(i) name two pairs of equiangular triangles.

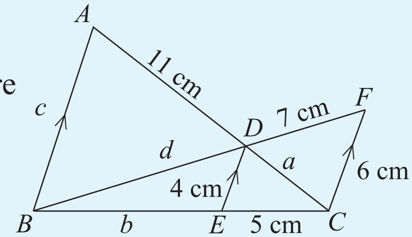
(ii) if $BD = 9$ cm, $GC = 6$ cm, $AG = 12$ cm and $GE = 2$ cm, then find the lengths of GB and AF.



4. According to the information given in the figure

(i) name three pairs of equiangular triangles.

(ii) find the lengths of the sides represented by a, b, c and d.



Next we investigate the converse of the above theorem. That is, to find whether the statement "if the sides of two triangles are proportional then those triangles are equiangular" is true .This converse is true and we can state this result as a theorem.

Also, if the three sides of a triangle are proportional to the three sides of another triangle then the two triangles are said to be similar.

Do the following activity to understand this result further.

Activity

- Construct the triangle ABC with $AB = 2.5$ cm, $BC = 3$ cm and $AC = 3.5$ cm.
- Construct the triangle PQR with $PQ = 5$ cm, $QR = 6$ cm and $PR = 7$ cm.
- Observe the relationship between the values $\frac{AB}{PQ}, \frac{BC}{QR}, \frac{AC}{PR}$.
- Measure the angles of each triangles separately.
- What type of triangles are ABC and PQR?

Through this activity you may have observed that the corresponding sides of the two triangles are proportional and also that the angles of ABC are equal to the angles of PQR.

This result can be expressed as the converse theorem of the theorem which we learnt about equiangular triangles earlier.

Theorem: If the three sides of a triangle are proportional to the three sides of another triangle, then the two triangles are equiangular.

Example 1

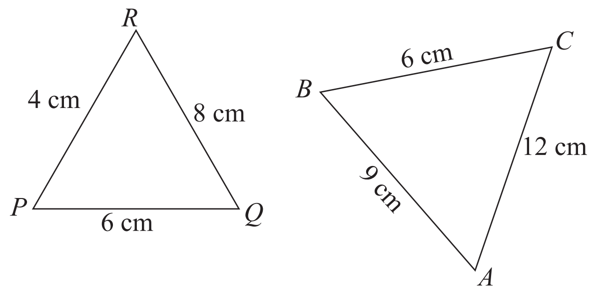
According to the lengths of the sides given in the figure, show with reasons that the triangles *ABC* and *PQR* are equiangular. Name the equal angles.

By writing the ratios of the sides according to the given lengths,

(i) $\dfrac{PQ}{AB} = \dfrac{6}{9} = \dfrac{2}{3}$

(ii) $\dfrac{RQ}{CA} = \dfrac{8}{12} = \dfrac{2}{3}$

(iii) $\dfrac{PR}{BC} = \dfrac{4}{6} = \dfrac{2}{3}$



Since these ratios are equal, according to the converse theorem, the triangles *PQR* and *ABC* are equiangular.

In the triangle *PQR*, $\hat{R}$ is the angle opposite *PQ*

$\hat{Q}$ is the angle opposite *PR*

$\hat{P}$ is the angle opposite *QR*

In the triangle *ABC*, $\hat{C}$ is the angle opposite *AB*

$\hat{A}$ is the angle opposite *BC*

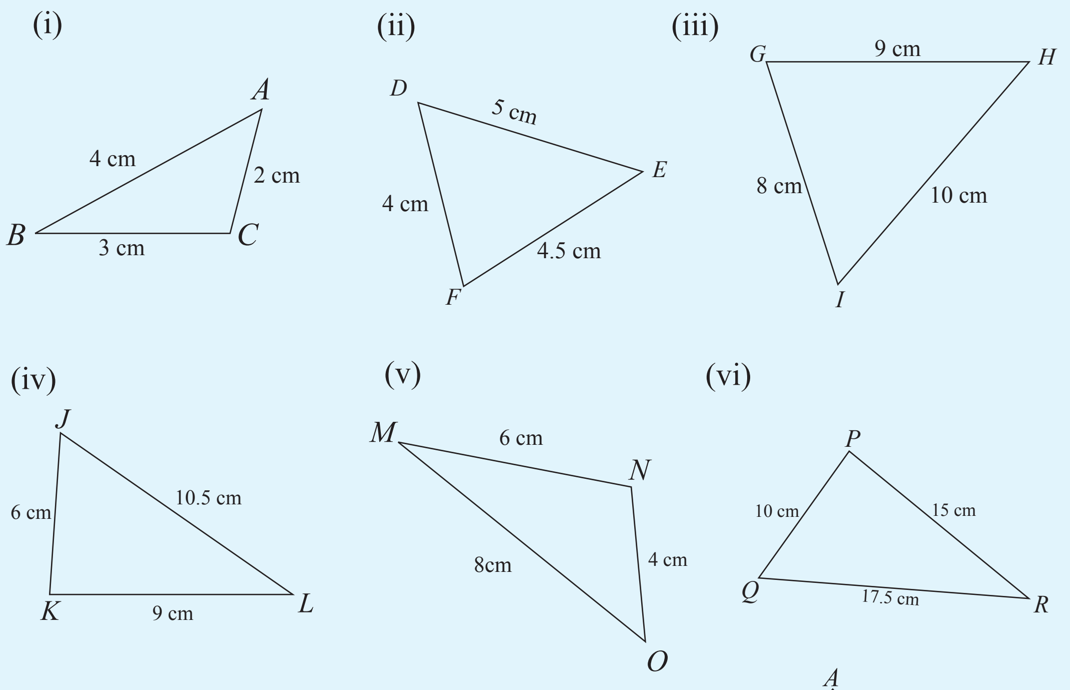
$\hat{B}$ is the angle opposite *AC*

$\therefore\ \hat{P} = \hat{B},\ \hat{Q} = \hat{A},\ \hat{R} = \hat{C}$

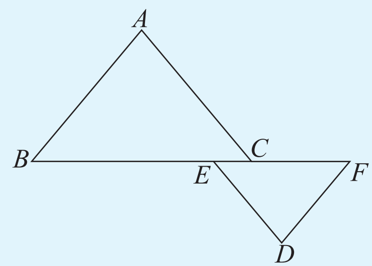
Do the following exercise using the converse theorem " if corresponding sides are proportional then the triangles are equiangular"

Exercise 14.5

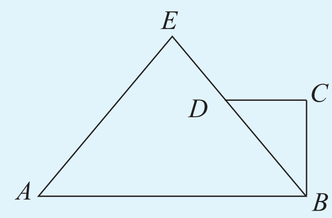
1. From the sketched triangles with measurements given below, choose three pairs of equiangular triangles.



2. In the given figure, $\frac{AB}{EF}=\frac{AC}{ED}=\frac{BC}{DF}$. Name an angle which is equal to each of $B\hat{A}C$, $A\hat{B}C$ and $A\hat{C}B$.



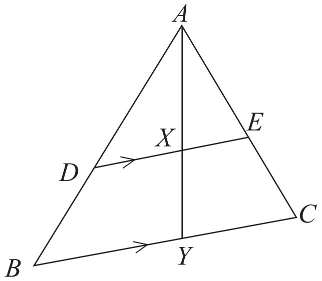
3. In the given figure, $AB = 20$ cm, $BC = 6$ cm, $CD = 4$ cm, $DB = 8$ cm, $DE = 2$ cm and $AE = 15$ cm. $AB//DC$. Also CD produced meets AE at F. Find the length of AF.



14.5 Proving riders using the theorem of equiangular triangles

Let us learn how to prove riders by using the theorems learnt so far appropriately. For that study the following example.

Example 1



In triangle ABC, the points D and E lie on the sides AB and AC such that $DE//BC$. The line AY cuts DE at X and BC at Y.

Prove that

(i) $\dfrac{XE}{YC} = \dfrac{AX}{AY}$

(ii) $\dfrac{XE}{YC} = \dfrac{DX}{BY}$

Proof : (i) In triangles, AXE and AYC in the figure ;

$A\hat{X}E = A\hat{Y}C$ (corresponding angles, $XE//YC$)

$A\hat{E}X = A\hat{C}Y$ (corresponding angles, $XE//YC$)

$\hat{A}$ is common to both triangles.

$\therefore$ AXE and AYC are equiangular triangles.

$\therefore$ corresponding sides are proportional.

Then; $\dfrac{AX}{AY} = \dfrac{XE}{YC}$ (By theorem)

(ii) In triangles, ADX and ABY in the figure,

$A\hat{D}X = A\hat{B}Y$ (corresponding angles, $DX//BY$)

$A\hat{X}D = A\hat{Y}B$ (corresponding angles, $DX//BY$)

$\hat{A}$ is common to both triangles.

$\therefore$ ADX and ABY are equiangular triangles.

$\therefore$ corresponding sides are proportional.

$\therefore \quad \dfrac{AX}{AY} = \dfrac{DX}{BY}$

But $\dfrac{AX}{AY} = \dfrac{XE}{YC}$ (proved)

$\therefore \quad \dfrac{XE}{YC} = \dfrac{DX}{BY}$

Now do the following exercise.

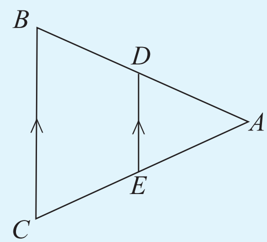
Exercise14.6

1. Based on the information given in the figure,

(i) show that triangles ADE and ABC are equiangular.

(ii) prove that $\dfrac{AD}{AB} = \dfrac{DE}{BC}$.

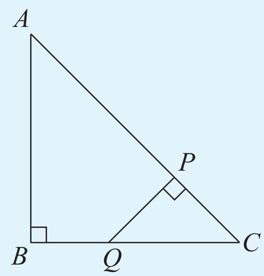
(iii) prove that $\dfrac{AE}{ED} = \dfrac{AC}{CB}$.



2. Based on the information given in the figure, prove that

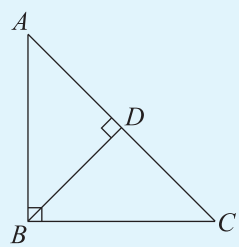
(i) the triangles ABC and PQC are equiangular.

(ii) $\dfrac{QC}{AC} = \dfrac{PQ}{AB} = \dfrac{PC}{BC}$.



3. In triangle ABC, $\hat{B}$ is a right angle. BD is the perpendicular drawn from B to AC. Prove that,

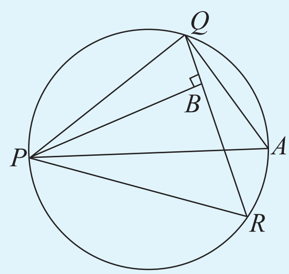
(i) $AB^2 = AD.\ AC.$



4. PA is a diameter of the circumcircle of triangle PQR. The line PB is the perpendicular drawn from P to QR. Prove that

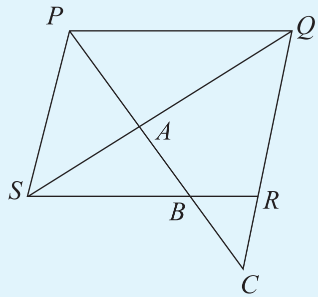
(i) the triangles PQA and PBR are equiangular

(ii) $\dfrac{PQ}{PB} = \dfrac{PA}{PR}$.



5. In parallelogram $PQRS$, the bisector of $Q\hat{P}S$ meets the diagonal QS at A, the side SR at B and QR produced at C.

Prove that $\dfrac{PQ}{PS} = \dfrac{PC}{PB}$.



6. In triangle ABC, the point P lies on AB and the point Q lies on AC such that $A\hat{P}Q = A\hat{C}B$. Prove that $AP.AB = AQ.AC$

7. The vertices of a triangle ABC lie on a circle. The bisector of $B\hat{A}C$ intersects the side BC at Q and the circle at P. Prove that $AC : AP = AQ : AB$

8. In triangle ABC, the bisector of $B\hat{A}C$ meets BC at D. The point X lies on AD produced such that $CX = CD$. Prove that,

(i) the triangles ACX and ABD are equiangular.

(ii) $\dfrac{AB}{AC} = \dfrac{BD}{DC}$

Miscellaneous Exercise

1. In the rectangle $ABCD$, E is on DC such that $A\hat{E}B = 90°$. Prove that ADE, AEB and EBC are similar triangles.

2. In the triangle ABC, $\hat{B}$ is a right angle. $AB = 5$ cm and $BC = 2$ cm. The perpendicular bisector of AC intersects the side AB at Q. Show that the length of $AQ = 2.9$ cm.

3. In triangle ABC, PQ drawn parallel to BC meets the side AB at P and the side AC at Q. The lines CP and BQ intersect each other at S. SR drawn parallel to AB meets BC at R.

Prove that $\dfrac{BR}{RC} = \dfrac{AQ}{AC}$.

15

Data Representation and Interpretation

By studying this lesson, you will be able to

- find the limits and boundaries of the intervals of a frequency distribution
- draw the relevant histogram
- draw the relevant frequency polygon
- draw the relevant cumulative frequency curve and find the inter quartile range from that curve

Limits and boundaries of class intervals

The following is data of the heights (to the nearest centimetre) of 30 students

137, 135, 141, 147, 151, 135, 137, 143, 144, 145
140, 134, 141, 140, 153, 144, 133, 138, 155, 130
136, 137, 142, 143, 145, 143, 154, 146, 148, 158

We know that the range is the value which is obtained by subtracting the least value of the data from the highest value of the data. Therefore,

$$\text{The range of the data} = 158 - 130$$
$$= 28$$

To facilitate interpretation, a group of data is often represented as a frequency distribution. We know that if the range of the data is large, then the data is divided into class intervals. Such a frequency distribution is called a **grouped frequency distrbution**. In such frequency distributions, generally, the number of class intervals should be between 5 and 10. The size of a class interval is determined by dividing the range of the frequency distribution by the number of class intervals and taking the least integer greater than that value .

For example let us group the above data into 6 class intervals. To find the size of a class interval, first divide 28 (the range) by 6 (number of class intervals).

$$\text{then, we obtain } \frac{28}{6} \approx 4.66 .$$

Hence the size of each interval should be 5, which is the least integer greater than 4.66.

Next the first class interval needs to be selected. Since the lowest value of the data is 130, the first class interval should start from 130.

The following are two different grouped frequency distributions from the given data.

Class intervals	Frequency
130 -135	3
135 - 140	7
140 - 145	10
145 - 150	5
150 - 155	3
155 - 160	2

First grouped distribution

Class intervals	Frequency
130 -134	3
135 - 139	7
140 - 144	10
145 - 149	5
150 - 154	3
155 - 159	2

Second grouped distribution

Consider the first grouped distribution. As an example, the 130 - 135 class interval represents the heights greater than or equal to 130 and less than 135. The second class interval 135 - 140 represents the heights greater than or equal to 135 and less than 140. The other class intervals can be described similarly.

Now consider the second grouped distribution. As an example, the 130 - 134 class interval represents the heights greater than or equal to 130 and less than or equal to 134.

Let us observe another difference between these two distributions. In the first distribution there are no gaps between the class intervals. For example, the class interval 135 - 140 starts from the upper limit 135 of the previous class interval 130 - 135. So there is a common limit for these class intervals. But this is not so in the second distribution. For example 134 is the upper limit of the class interval 130-134, but the next class interval starts from 135. The gap between these two limits is 1.

In the next section of this lesson we hope to learn how to draw histograms. To draw a histogram, there should not be these kinds of gaps. Therefore we must change the second distribution appropriately. This change can be done by introducing a common boundary for the class intervals. That common boundary can be determined easily.
For example, in the second distribution, 134.5 is taken as the boundary of the class intervals 130 - 134 and 135 - 139, which is the exact middle of the upper limit (134) of the class interval 130 - 134 and the lower limit (135) of the class interval 135 - 139.The new distribution constructed in this way is given below.

Class intervals with boundaries	Frequency
129.5 - 134.5	3
134.5 - 139.5	7
139.5 - 144.5	10
144.5 - 149.5	5
149.5 - 154.5	3
154.5 - 159.5	2

Here observe that, 0.5 is subtracted from the lower limits and 0.5 is added to the upper limits of every class interval of the original distribution.This rule is valid for the first and last class intervals too. The values 129.5 and 159.5 are obtained in that way. Also observe that, the size of the class interval of this new distribution is 5.

In the above, the first kind of distribution is simple. But practically, the second kind of distribution can be constructed easily. Both kinds of distributions can be found in statistics.

15.1 Histogram of a grouped frequency distribution

Let us consider how to draw the histogram of a grouped frequency distribution. A histogram is a graphical representation of a grouped frequency distribution. In a histogram, the frequencies of class intervals are represented by the heights of the rectangular columns which touch each other. Let us consider first, how to draw the histogram if the sizes of the class intervals are equal.(As in the example in the previous section).

When drawing the histogram, follow the steps given below.

- Mark the boundaries of the calss intervals on the horizontal axis drawn to an appropriate scale.
- Along the vertical axis, on an appropriate scale, draw the columns such that the height of the column on a class interval is the corresponding frequency.

By considering the following example, let us observe how to draw the histogram.

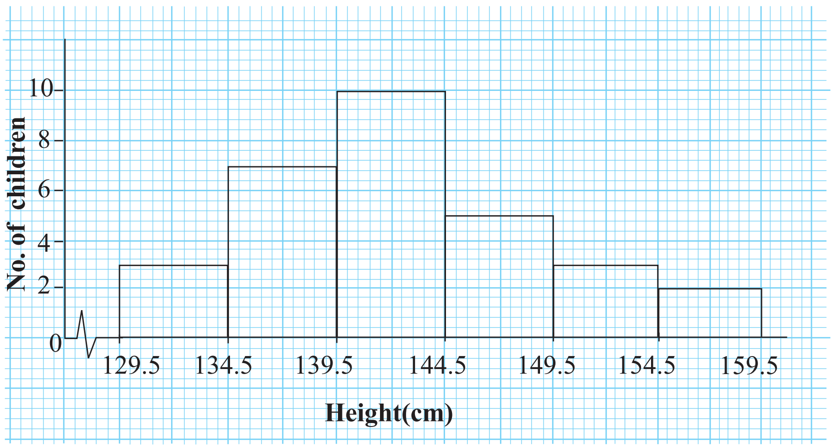
Example 1

Draw the histogram of the grouped frequency distribution prepared in the previous section.

For this, let us consider the second frequency distribution.

Intervals with boundaries	Frequency
129.5 - 134.5	3
134.5 - 139.5	7
139.5 - 144.5	10
144.5 - 149.5	5
149.5 - 154.5	3
154.5 - 159.5	2

The relevant histogram is given below.
Two small squares along the horizontal axis represents 1 centimetre.
Five small squares along the vertical axis represents two children.



Observe here that the columns touch each other.

Note: Since the data starts from 129.5, it is not necessary to show the class intervals from 0 to 129.5 in the histogram. The mark at the beginning of the *x* axis indicates that the axis has been shrunk in this region.

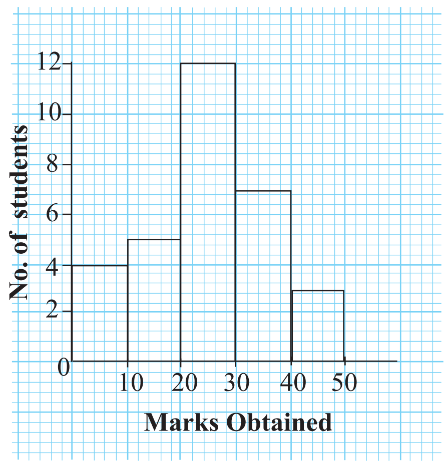
Example 2

The following is a frequency distribution of the Mathematics marks of students taken from a School Based Assessment.

Class intervals (Marks Obtained)	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
Frequency (Number of children)	4	5	12	7	3

As an example, the interval 0 - 10 represents the marks greater than or equal to 0 and less than 10. The other class intervals are defined similarly. Draw the histogram of the frequency distribution.

In this frequency distribution, the first class interval ends at 10 and next class interval starts at 10.This histogram can be drawn easily.



Now let us consider how to draw a histogram of a frequency distribution with different sized class intervals.

Example 3

A frequency distribution based on the marks of 40 students in a term test is given below.

Class intervals (Marks Obtained)	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 70	70 - 100
Frequency (Number of students)	2	4	6	9	5	8	6

If you observe the class intervals, you can see that the sizes of the class intervals are not the same. The size of each of the first 5 class intervals is 10 and the sizes of the next two class intervals are 20 and 30 respectively. Another important feature in a histogram is that the areas of the columns are proportional to the relevant frequencies.

Therefore if the sizes of the class intervals are equal, then the frequencies are proportional to the heights of the columns. Hence in the above examples 1 and 2, the frequency can be identified directly with the height of the column. But in this example, since the sizes of the class intervals are not the same, the frequency cannot be identified directly with the height of the column. The heights of the columns should be obtained such that the areas of the columns are proportional to the frequences. This can be done as follows.

In this frequency distribution, the size of all but the class intervals 50 - 70 and 70 - 100 is 10. The size of the class interval 50 - 70 is 20 and the size of the class interval 70 - 100 is 30.

Hence the size of the smallest class interval is 10 and the size of the class interval 50 - 70 is two times that. Since the area of the column which represents the frequency of the class interval should be proportional to the frequency,

$$\text{Height of the column} = \frac{\text{frequency}}{2}$$

$\therefore$ The height the column of the 50 - 70 class interval $= \frac{8}{2}$

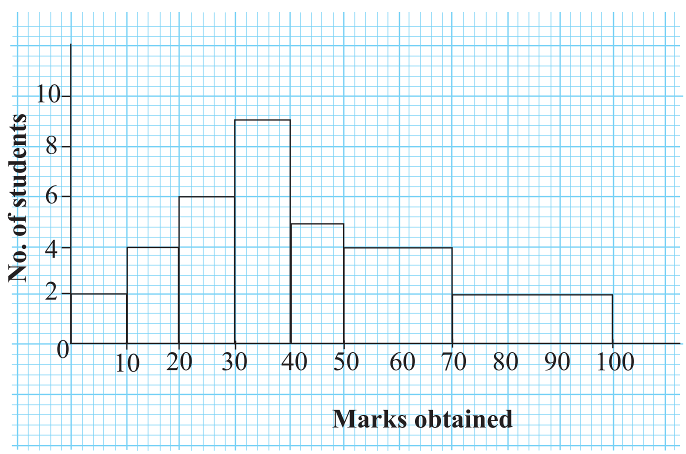
$= 4$

The size of the class interval 70 - 100 is three times the size of the smallest class interval.

$\therefore$ The height the column of the 70 - 100 class interval $= \frac{6}{3}$

$= 2$

After these calculations, the histogram can be drawn as follows.



Exercise 15.1

1. The frequency distribution prepared from the data collected by a weather forecasting center in a certain area is given below. Illustrate this information in a histogram.

Rainfall in a week in mm.	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
Number of weeks	5	6	15	10	7	5	4

2. The frequency distribution of the number of books borrowed from a school library in the year 2015 is given below. Illustrate this information in a histogram.

Class intervals (Number of books issued)	25 - 29	30 - 34	35 - 39	40 - 44	45 - 49	50 - 54
Frequency (Number of days)	5	10	20	15	10	7

3. The frequency distribution prepared from the data obtained by measuring the circumference of teak trees in a forest plantation is given below. Illustrate this information in a histogram.

Circumference of a tree (cm)	30 - 34	35 - 39	40 - 44	45 - 49	50 - 54	55 - 59
Number of trees	6	8	9	15	24	21

4. The frequency distribution prepared from the data on the daily consumption of water provieded through rural water project to 60 houses is given below. Illustrate this information in a histogram.

Home usage of water (Litres)	8 - 12	13 - 17	18 - 22	23 - 27	28 - 32	33 - 37	38 - 42
Number of houses	4	6	15	15	10	7	3

5. The information on the monthly electricity consumption of 75 houses in January 2015 is shown in the table given below. Illustrate this information in a histogram.

Class intervals (Units of electricity)	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 100
Frequency (Number of houses)	10	11	14	16	12	12

6. The following frequency distribution shows the information on the number of telephone calls and the time duration of each telephone call on a certain day at a communication centre.

Time duration of one telephone call (seconds)	30 - 45	45 - 60	60 - 75	75 - 90	90 - 120
Number of calls	8	9	12	16	8

15.2 Frequency polygon

A frequency polygon is a graphical representation of grouped data similar to a histogram. There are two methods to construct frequency polygons.

- From the histogram of the frequency distribution
- From the mid-values and frequencies of the class intervals

In the following example, let us consider first, how to construct a frequency polygon from the histogram.

Example 1

Let us consider a frequency distribution used in a previous example.

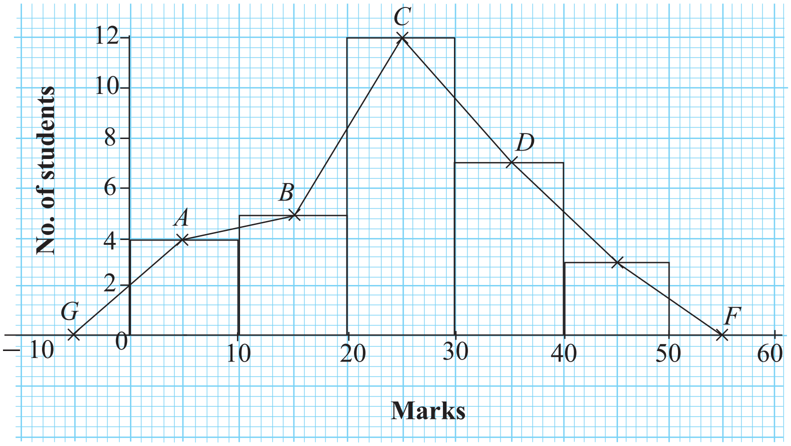
Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
Number of students	4	5	12	7	3

(i) First draw the histogram relevant to the given information

(ii) In each column in the histogram, mark the sign "×" at the middle of the top of the column. (See the following figure; the signs "×" denoted as A, B, C, D and E)

(iii) Join these "×"signs by the line segments as shown in the figure.

(iv) On the horizontal axis, mark half the length of the class interval to the right of the last class interval and to the left of the first class interval (hence 5 units). Join $E\,F$ and $A\,G$.



Now you get the polygon *ABCDEFG*. This polygon is called the frequency polygon of the frequency distribution. If you observe carefully, you can see that the area of the frequency polygon is equal to the sum of the areas of the columns of the histogram.

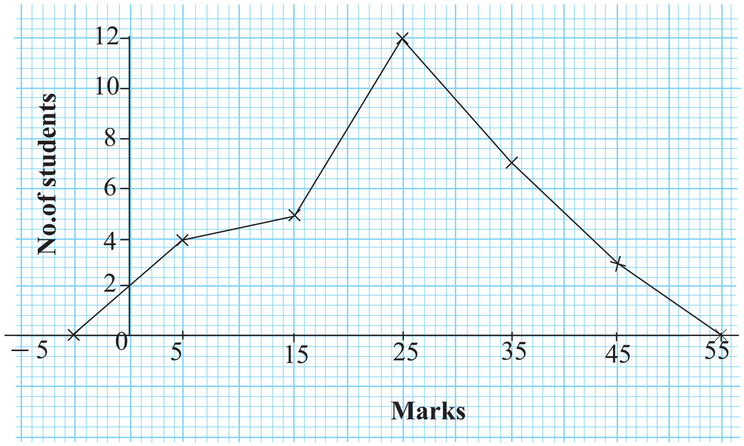
To draw a frequency polygon, it is not necessary to draw the histogram first. The frequency polygon can be drawn directly by using the mid-values of the class intervals and using the relevant frequencies. The following example shows how to draw the frequency polygon using this method.

Example 2

From the given frequency distribution, prepare a table with the mid-values of the class intervals to draw the frequency polygon.

Class interval	Mid-value	Frequency
0 - 10	5	4
10 - 20	15	5
20 - 30	25	12
30 - 40	35	7
40 - 50	45	3

Mark the mid-values of the class intervals along the horizontal axis and mark the frequencies along the vertical axis. Then mark the corresponding points. The frequency polygon can be obtained by joining those points respectively by line segments.



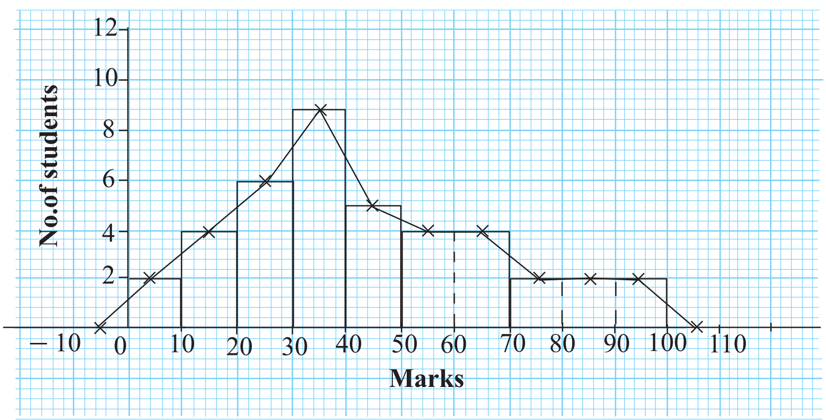
Let us consider next, how to draw a frequency polygon with unequal sized class intervals.

Example 3

Class intervals (Obtained marks)	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 70	70 - 100
Frequency (Number of students)	2	4	6	9	5	8	6

Let us draw the frequency polygon for the above frequency distribution with unequal sized class intervals.

The relevant frequency polygon is given below.



Here, the class interval 50 - 60 of size 20 is divided in to two class intervals of size 10 and as the relevant frequency, the frequency of the mid-point of each class interval is considered. Similarly the class interval 70 - 100 of size 30 is divided in to three class intervals of size 10 and as the relevant frequency, the frequency of the mid-point of each class interval is considered. Also in this case, observe that the area of the frequency polygon is equal to the sum of the areas of columns of the histogram.

Exercise 15.2

1. The frequency distribution prepared from the data collected by measuring the masses of students who attended a medical clinic of a school is given below.

Mass of a student (kg)	30 - 35	35 - 40	40 - 45	45 - 50	50 - 55
Number of students	8	10	15	7	15

(i) Illustrate this information in a histogram.
(ii) Draw the frequency polygon on this histogram.

2. The following is a frequency distribution prepared from the data collected through a test conducted to determine the life time of bulbs manufactured by a company.

Class intervals (No. of hours the bulb lighted)	100 - 300	300 - 400	400 - 500	500 - 600	600 - 700	700 - 800
Frequency (Number of bulbs)	12	10	20	25	15	12

(i) Illustrate this information in a histogram.
(ii) Draw the frequency polygon on this histogram.

3. Information on the mass of the members of a sport club is given in the following table.

Mass (kg)	60 - 65	65 - 70	70 - 75	75 - 80	80 - 85
Number of members	10	15	6	4	2

(i) From this information prepare a table with the mid-values of the class intervals.
(ii) Draw the frequency polygon using the mid-values of the class intervals.

4. The following is a frequency distribution prepared from the Mathematics marks of grade11 students in a school.

Class intervals (Marks)	0 - 30	30 - 40	40 - 50	50 - 60	60 - 100
Frequency (Number of students)	6	5	10	7	12

(i) Draw the histogram of this information and use it to draw the frequency polygon.

5. The table given below is prepared from the information of a communication centre on the number of telephone calls taken on a certain day and the time duration of those calls.

Time duration of one telephone call (seconds)	1 - 4	4 - 7	7 - 10	10 - 13	13 - 16
Number of calls	3	9	20	12	6

(i) Draw the histogram of this frequency distribution.
(ii) Use this histogram to draw the frequency polygon.

15.3 Cumulative frequency curve of a grouped frequency distribution

This is a another method of representing data of a frequency distribution graphically.

Let us see how to draw the cumulative frequency curve by considering the following example

Example 1

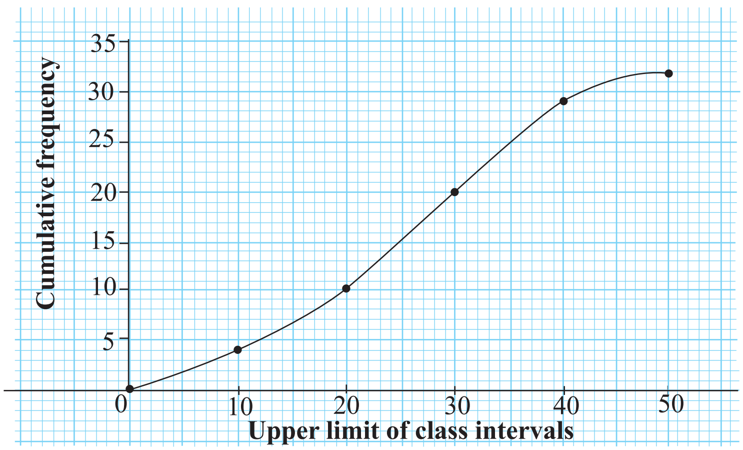
The following is a frequency distribution of the marks in Mathematics of 32 students in a class. Let us draw its cumulative frequency curve.

Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50
Number of students	4	6	10	9	3

Let us construct a cumulative frequency table from the above table.

Class intervals	Frequency	Cumulative frequency
0 - 10	4	4
10 - 20	6	10
20 - 30	10	20
30 - 40	9	29
40 - 50	3	32

The word "cumulative" means "increasing by addition". For example, in the above table, the cumulative frequency relevant to the class interval 20-30 is the sum of all frequencies less than in the intervals up to the class mark 30. (In other words no. of students whose marks are less than 30) That is equal to 20. The cumulative frequency relevant to the class interval 40 - 50 is the number of students whose marks are less than 50. That is 32, the total number of students. After completing this table, to draw the cumulative frequency curve, all the points corresponding to the upper limit of each class interval via the relevant cumulative frequency should be marked. These points should be joined smoothly, as shown below.



15.4 Quartiles and interquartile range of a frequency distribution

In previous sections, we learnt how to construct a histogram, frequency polygon and cumulative frequency curve from a group of data. These are useful to get an idea about how far the data disperses centrally. For example, just by looking at the histogram, the modal class of a grouped frequency distribution can be decided. Similarly, an idea on how far the data disperses symmetrically can also be obtained. In this section, we hope to learn about quartiles and the interquartile range of a group of data. From this we can get some idea about the dispersion of data.

The first thing to do to find quartiles and the interquartile range is to organize the data in ascending order. After that the first quartile (Q_1), the second quartile (Q_2) and the third quartile (Q_3) can be found as follows.

Step **1**: First find the median of the data. This is the second quartile.

Step **2**: Find the median of the data to the left of the second quartile. This is the first quartile.

Step **3**: Find the median of the data to the right of the second quartile. This is the third quartile.

As an example, consider the following group of data which is in ascending order.

Example 1

5, 6, 6, 8, 11, 12, 12, 12, 13, 14, 14, 14, 17, 18, 20, 24, 25, 26, 30

Here,there are 19 data. The median of these is 14 (It is in the square)

5, 6, 6, 8, 11, 12, 12, 12, 13, $\boxed{14,}$ 14, 14, 17, 18, 20, 24, 25, 26, 30

Now consider the data left of the median.

5, 6, 6, 8, $\boxed{11,}$ 12, 12, 12, 13

The median of that part is 11. That is also in the square.
At last, consider the data right of the median. .

14, 14, 17, 18, $\boxed{20,}$ 24, 25, 26, 30

The median of that part is 20. That is also in a square.
Hence,

first quartile $= Q_1 = 11$
second quartile $= Q_2 = 14$
third quartile $= Q_3 = 20.$

Example 2

Let us find the quartiles of the following 18 data written in ascending order .

2, 2, 3, 6, 6, 6, 7, 8, 8, 11, 11, 12, 12, 15, 15, 16, 17, 20

2, 2, 3, 6, 6, 6, 7, 8, $\boxed{8,\ 11,}$ 11, 12, 12, 15, 15, 16, 17, 20

Its median is the mean of the 8 and 11 in the rectangle.

Hence,

$$Q_2 = \frac{8 + 11}{2} = 9.5$$

The left side of the median is:

2, 2, 3, 6, $\boxed{6,}$ 6, 7, 8, 8

The median of that is 6. It is in the square.
Hence $Q_1 = 6$.

At last, the right side of the median is:

11, 11, 12, 12, $\boxed{15,}$ 15, 16, 17, 20

The median of that is 15. It is in the square.
Hence $Q_3 = 15$.

Example 3

There are 17 data in the following data string. Find the quartiles.

102, 104, 104, 105, 107, 107, 107, 108, 112, 112, 113, 115, 115, 119, 120, 125, 126

The heads of the arrows indicate the quartiles which are calculate according to the above steps.

102, 104, 104, $\boxed{105,\ 107,}$ 107, 107, 108, $\boxed{112,}$ 112, 113, 115, $\boxed{115,\ 119,}$ 120, 125, 126

$$Q_1 = \frac{105 + 107}{2} = 106$$

$$Q_2 = 112$$

$$Q_3 = \frac{115 + 119}{2} = 117$$

Example 4

There are 16 data in the following data string. Observe how to calculate the quartiles which are indicated by the heads of arrows.

$$21, 23, 25, \boxed{25, 26,} 28, 28, \boxed{30, 30,} 34, 34, \boxed{35, 37,} 37, 40, 42$$

Hence, $Q_1 = \frac{25 + 26}{2} = 25.5$, $Q_2 = \frac{30 + 30}{2} = 30$, $Q_3 = \frac{35 + 37}{2} = 36$.

In statistics, there are several methods to find quartiles of a data string. The method described here is the most practical and the most convenient method.

Another method to calculate quartiles is, finding the locations of the quartiles using the formulae

Q_1 in the $\frac{1}{4}$ (n + 1) position, Q_2 in the $\frac{2}{4}$ (n + 1) position, Q_3 in the $\frac{3}{4}$ (n + 1) position.

For example, consider the data string 4 6 7 8 15 18 20.

In this data string, according to the formulae,

Q_1 is located at the place $\frac{1}{4}(7 + 1) = 2$. Hence $Q_1 = 6$.

Q_2 is located at the place $\frac{2}{4}(7 + 1) = 4$. Hence $Q_2 = 8$.

Q_3 is located at the place $\frac{3}{4}(7 + 1) = 6$. Hence $Q_3 = 18$.

As another example, consider the data string 9 12 18 20 21 23 24 26.
In this data string, according to the formulae,

Q_1 is located at the place $\frac{1}{4}(8+1) = 2.25$. Hence $Q_1 = 12 + \frac{1}{4}(18-12) = 13.5$

Q_2 is located at the place $\frac{2}{4}(8+1) = 4.5$. Hence $Q_2 = \frac{20+21}{2} = 20.5$

Q_3 is located at the place $\frac{3}{4}(8+1) = 6.75$. Hence $Q_3 = 23 + \frac{3}{4}(24-23) = 23.75$

Here, when using different methods from each other, different answers with small deviations may be obtained. But this is not a problem since in methods of statistics it is expected to get approximate values.

The interquartile range of a group of data is the value obtained by subtracting the first quartile from the third quartile.

Hence $\boxed{\textbf{Interquartile range} = Q_3 - Q_1}$

Exercise 15.4

1. The following are ages of 17 workers in a work place prepared in ascending order.

21, 22, 23, 24, 25, 27, 27, 30, 34, 35, 40, 41, 42, 44, 46, 47, 50

For this group of data, find the following.

(i) Median
(ii) First quartile
(iii) Third quartile
(iv) Interquartile range.

2. The number of members in the families of the students in a class is given below.

7, 6, 4, 3, 8, 5, 5, 4, 3, 6, 4, 6, 7, 10, 5

Prepare this group of data in ascending order and find the following.

(i) Median
(ii) First quartile
(iii) Third quartile
(iv) Interquartile range.

3. The information on the electricity consumption of 32 shops in a town during a day in the year 2015 of 32 shops in a town is given below.

No. of units	2	3	4	5	6	7	8	10
No. of shops	5	2	6	6	7	2	3	1

For this group of data, find the following
(i) Median
(ii) First quartile
(iii) Third quartile
(iv) Interquartile range.

More on Interquartile Range

In this section, we hope to learn how to find quartiles and the interquartile range of grouped data. Here we describe only, how to find it using the cumulative frequency curve. Let us consider how to find quartiles and the interquartile range of grouped data from the following example.

Example 1

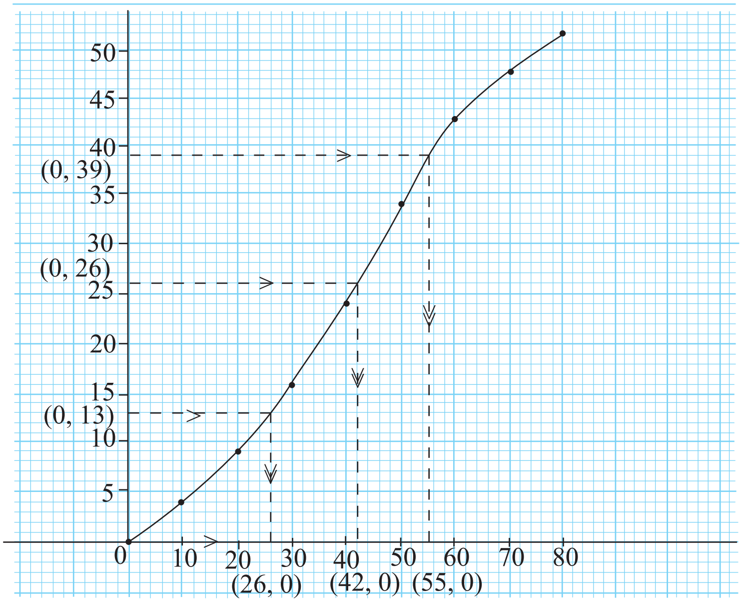
The following is a frequency distribution prepared from the Mathematics marks of a group of students in grade 11. Let us draw the cumulative frequency curve for this frequency distribution.

Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60	60 - 70	70 - 80
No. of students	4	5	7	8	10	9	5	4

Let us construct a table from this data to draw the cumulative frequency curve.

Class intervals	Frequency	Cumulative frequency
0 - 10	4	4
10 - 20	5	9
20 - 30	7	16
30 - 40	8	24
40 - 50	10	34
50 - 60	9	43
60 - 70	5	48
70 - 80	4	52

Let us draw the cumulative frequency curve as learnt in section 15.3.



Now let us pay attention to the vertical lines and the horizontal lines in the above figure of the cumulative frequency curve.
Here the total number of data is 52. That is, the sum of the frequencies is 52. Let us first find, the locations of the first, second and third quartiles of those 52 data.

Note: When finding the quartiles from the cumulative frequency curve, there is no need to do it as in setion 15.5. Since the number of data is large (More than 30 data values is considered as a large number of data), it is sufficient to locate the positions of $\frac{1}{4}$ of the frequencies, $\frac{1}{2}$ of the frequencies and $\frac{3}{4}$ of the frequencies.

When the cumulative frequency is increasing then the first quartile is located at the position of $\frac{1}{4}$ th of the total frequency. Hence,

$$\text{the position of } Q_1 = \frac{1}{4} \times 52^{\text{th}} \text{ position} = 13^{\text{th}} \text{ position}$$

$$\text{the position of } Q_2 = \frac{1}{2} \times 52^{\text{th}} \text{ position} = 26^{\text{th}} \text{ position}$$

$$\text{the position of } Q_3 = \frac{3}{4} \times 52^{\text{th}} \text{ position} = 39^{\text{th}} \text{ position}$$

Now it is needed to find the data corresponding to the points 13, 26 and 39 (frequencies) of the vertical axis. The necessary lines are illustrated in the figure. As an example the first quartile can be found as follows.

Since the first quartile is located at the 13^{th} position, a horizontal line is drawn from the point 13 on the vertical axis until it meets the curve. From the point that line meets the curve, a vertical line is drawn until it meets the horizontal axis. The value of that point on the horizontal axis is the first quartile.

If we find the quartiles for the given example, then we get $Q_1 = 26$, $Q_2 = 42$ and $Q_3 = 55$.

Hence, the interquartile range $= Q_3 - Q_1 = 55 - 26 = 29$.

As an example, if the total frequency of a frequency distribution is 51 then the first, the second and the third quartiles are in the

$$\frac{1}{4} \times 51 = 12.75^{th} \text{ position}$$

$$\frac{1}{2} \times 51 = 25.5^{th} \text{ position}$$

$$\frac{3}{4} \times 51 = 38.25^{th} \text{ position respectively.}$$

Hence the quartiles can be found by considering the values on the x-axis corresponding to the values 12.75, 25.5 and 38.25 (or the appropriate values rounded to the scale of the graph) on the vertical axis.

Exercise 15.5

1. The following is the information on the leave taken by the employees of a certain office in the year 2015.

Number of days	0 - 4	4 - 8	8 - 12	12 - 16	16 - 20	20 - 24
Number of employees	10	18	11	8	5	4

(i) Construct the cumulative frequency table with the above information.
(ii) Draw the cumulative frequency curve from the table.
(iii) From the cumulative frequency curve find the following.
 (*a*) Median of the leave taken by the employees.
 (*b*) Interquartile range of the data.

2. The following is a table of marks obtained in a monthly test of grade 11 students for science.

Class interval of marks	0 - 15	15 - 30	30 - 45	45 - 60	60 - 75	75 - 90
No. of students	6	8	12	20	10	4

(i) From the data of the table, construct a cumulative frequency table.
(ii) Draw the cumulative frequency curve.
(iii) From the cumulative frequency curve find the following.
  (a) First quartile
  (b) Second quartile
  (c) Third quartile.
(iv) Find the interquartile range of the marks.

3. The following is information on the salaries of employees of a garment factory, in the month of January 2015. Using this information, draw the cumulative frequency curve. From that curve, find the median of salary of an employee and find the interquartile range of the salaries.

Monthly salary of an employee (Rupees) (Class interval)	20000 - 20500	20500 - 21000	21000 - 21500	21500 - 22000	22000 - 22500	22500 - 23000	23000 - 23500	23500 - 24000
No. of employees	8	10	15	18	25	12	9	7

Miscellaneous Exercise

1. The following table is prepared based on the monthly charges for the consumption of electricity of houses in a housing scheme.

Monthly charges (Rupees)	0 - 200	200 - 400	400 - 600	600 - 800	800 - 1000
No. of houses	8	14	24	12	6

(i) From this information, construct a cumulative frequency table.
(ii) Draw the cumulative frequency curve.
(iii) Find the median.
(iv) Find the interquartile range.

2. The following frequency distribution is prepared from the information on the ages of the staff of a certain office.

Age (years)	20 - 25	25 - 30	30 - 35	35 - 40	40 - 45	45 - 50	50 - 55	55 - 60
Number of employees	8	12	14	18	16	6	2	2

For the given grouped frequency distribution,

(i) draw the histogram

(ii) draw the frequency polygon.

(iii) draw the cumulative frequency curve.

(iv) Find the interquartile range from the cumulative frequency curve.

3. The following table is prepared from information on the water consumption of 100 houses in a housing scheme during a certain month.

No of units	20 - 29	30 - 39	40 - 49	50 - 59	60 - 69	70 - 79
No of houses	2	8	35	40	10	5

(i) From this information, draw the histogram and the frequency polygon.

(ii) Construct a cumulative frequency table.

(iii) From that table, draw the cumulative frequency curve.

(iv) Find the interquartile range of this information.

16 Geometric progressions

By studying this lesson you will be able to,

- identify whether a given number sequence is a geometric progression,
- use the formula for the n^{th} term of a geometric progression,
- use the formula for the sum of the first n terms of a geometric progression and
- solve problems related to geometric progressions.

16.1 Geometric Progressions

Let us first recall what we learnt in grade 10 about arithmetic progressions. Given below is an arithmetic progression.

5, 7, 9, 11, ...

In this sequence, by adding the constant value 2 to any term, the term next to it is obtained. We called this constant value the common difference of the arithmetic progression.

Now, careful observe the sequence given below.

3, 6, 12, 24, 48, 96, ...

In this sequence, the first term is 3. The second term is obtained by multiplying the first term by 2, third term is obtained by multiplying the second term by 2. The sequence continues in this manner, where, each term multiplied by 2 is the next term. In other words, if we divide any term other than the first term, by the previous term we get the constant value 2.

A sequence that yields a fixed value when any term, other than the first term, is divided by the previous term is called a **geometric progression**. The fixed value, by which each term is multiplied to obtain the next term, is called the **common ratio** of the geometric progression. The common ratio of the above geometric progression is 2.

Given a number sequence, one can check whether it is a geometric progression by doing the following test. Note down the value obtained by dividing the second term by the first term. Similarly, note down the values obtained by dividing the third term by the second term, fourth term by the third term and so forth. If the values noted down are all equal, then it is a geometric progression. When the noted values are all equal, this common value is the **common ratio**.

Example 1

Check whether the number sequence 2, 6, 18, 54, ... is a geometric progression.

$$\frac{6}{2} = 3, \qquad \frac{18}{6} = 3, \qquad \frac{54}{18} = 3$$

$$\therefore \quad \frac{6}{2} = \frac{18}{6} = \frac{54}{18} = 3$$

$\therefore$ The given sequence is a geometric progression. Moreover, the common ratio is 3.

Example 2

Check whether the number sequence 200, 100, 50, 20, ... is a geometric progression.

$$\frac{100}{200} = \frac{1}{2}, \qquad \frac{50}{100} = \frac{1}{2}, \qquad \frac{20}{50} = \frac{2}{5}$$

Because the ratios are not equal, this sequence is not a geometric progression.

Example 3

Check whether the number sequence 5, – 10, 20, – 40, 80, ... is a geometric progression.

$$\frac{-10}{5} = -2, \qquad \frac{20}{-10} = -2, \qquad \frac{-40}{20} = -2, \qquad \frac{80}{-40} = -2$$

$$\therefore \quad \frac{-10}{5} = \frac{20}{-10} = \frac{-40}{20} = \frac{80}{-40} = -2$$

$\therefore$ This is a geometric progression with common ratio – 2.

Example 4

The terms 4, x, 16 are consecutive terms of a geometric progression. What is the value of x?

$\frac{x}{4} = \frac{16}{x}$ because the terms are in a geometric progression. We can find the value of x by solving this equation.

If $\frac{x}{4} = \frac{16}{x}$, then $x^2 = 64$.

Thus $x^2 - 8^2 = 0$

$\therefore\ (x - 8)(x + 8) = 0$

$\therefore\ x = 8$ or $x = -8$

Let us check for each of the values we have obtained, whether the three terms 4, x, 16 are in a geometric progression.

When $x = 8$, the sequence 4, 8, 16 is a geometric progression with common ratio 2. When $x = -8$, the sequence 4, -8, 16 is a geometric progression with a common ratio -2.

Exercise 16.1

1. Select and write down the geometric progressions from the number sequences given below.

(a) 2, 4, 8, ... **(b)** $-6, -18, -54, \ldots$ **(c)** 64, 32, 16, 8, ...

(d) 5, 10, 30, 120, ... **(e)** $-2, 6, -18, 54, \ldots$ **(f)** 81, 27, 3, $\frac{1}{9}$, ...

(g) 0.0002, 0.002, 0.02, 0.2, ... **(h)** $\frac{1}{2}, \frac{1}{6}, \frac{1}{18}, \frac{1}{36}, \frac{1}{72}, \ldots$

16.2 n^{th} term of a Geometric Progression

Recall that you learnt, in Grade 10, that the n^{th} term of an arithmetic progression with first term a and common difference d is $T_n = a + (n - 1)d$.

Let us now consider how we can obtain an expression for the n^{th} term of a geometric progression. We will write "a" for the first term of the geometric progression and "r" for the common ratio. Moreover, we will denote the n^{th} term of the sequence by T_n.

Consider the geometric progression 2, 6, 18, 54, ... Here, the first term (a) is 2 and the common ratio (r) is 3.

Carefully observe that;

$$T_1 = 2 = 2 \times 1 = 2 \times 3^{1-1}$$
$$T_2 = 6 = 2 \times 3 = 2 \times 3^{2-1}$$
$$T_3 = 18 = 2 \times 3 \times 3 = 2 \times 3^{3-1}$$
$$T_4 = 54 = 2 \times 3 \times 3 \times 3 = 2 \times 3^{4-1}$$

The above expressions can be written in terms of the first term (a) and the common ratio (r), as

$$T_1 = 2 \times 3^0 = a \times r^{1-1}$$
$$T_2 = 2 \times 3^1 = a \times r^{2-1}$$
$$T_3 = 2 \times 3^2 = a \times r^{3-1}$$
$$T_4 = 2 \times 3^3 = a \times r^{4-1}$$

According to the observable pattern, the n^{th} term T_n can be written as $T_n = ar^{n-1}$.

> The n^{th} term of the geometric progression with first term $\boldsymbol{a}$ and common ratio $\boldsymbol{r}$ is given by
> $$\boldsymbol{T_n = ar^{n-1}}.$$

Example 1

Find the 5^{th} term of the geometric progression with first term 3 and common ratio 2.

$a = 3,\ r = 2, n = 5$

$$\begin{aligned} T_n &= ar^{n-1} \\ T_5 &= 3 \times 2^{5-1} \\ &= 3 \times 2^4 \\ &= 3 \times 16 \\ &= 48 \end{aligned}$$

Therefore, the 5^{th} term is 48.

Example 2

Find the fifth and the seventh terms of the geometric progression 81, 27, 9,...

$$a = 81$$
$$r = \frac{27}{81} = \frac{1}{3}$$
$$T_n = ar^{n-1}$$
$$\therefore \quad \begin{aligned} T_5 &= 81 \times \left(\frac{1}{3}\right)^{5-1} \\ &= 81 \times \left(\frac{1}{3}\right)^4 \\ &= 81 \times \frac{1}{81} \\ &= 1 \end{aligned}$$

$$\begin{aligned} T_7 &= 81 \times \left(\frac{1}{3}\right)^{7-1} \\ &= 81 \times \left(\frac{1}{3}\right)^6 \\ &= 81 \times \frac{1}{729} \\ &= \frac{1}{9} \end{aligned}$$

Thus, the fifth term is 1 and the seventh term is $\frac{1}{9}$.

Exercise 16.2

1. Find the 6^{th} term of the geometric progression with first term 5 and common ratio 2.
2. Find the 6^{th} and 8^{th} terms of the geometric progression with first term 4 and common ratio – 2.
3. Find the 4^{th} and 7^{th} terms of the geometric progression with first term – 2 and common ratio – 3.
4. Of a geometric progression, the first term is 1000 and the common ratio is $\frac{1}{5}$. Find the 6^{th} term.
5. Find the 6^{th} term of the geometric progression 0.0002, 0.002, 0.02,... .
6. Find the 5^{th} term of the geometric progression $\frac{3}{8}, \frac{3}{4}, 1\frac{1}{2},...$.
7. Find the 4^{th} term of the geometric progression 75, – 30, 12,... .
8. Find the 7^{th} term of the geometric progression 192, 96, 48,... .
9. Find the 9^{th} term of the geometric progression 0.6, 0.3, 0.15,...
10. Find the 10^{th} term of the geometric progression 8, 12, 18,...

16.3 Using the formula $T_n = ar^{n-1}$

When all but one from the first term (a), common ratio (r), n^{th} term T_n and n, of a geometric progression are given, the unknown value can be found by substituting the given values into $T_n = ar^{n-1}$.

consider at the following examples.

Example 1

Find the first term of the geometric progression with common ratio 3 and 4^{th} term 54.

$r = 3,\ n = 4,\ T_n = 54$

$$T_n = ar^{n-1}$$

$$\therefore\ T_4 = a \times (3)^{4-1}$$

$$\therefore\ 54 = a \times (3)^3$$

$$\therefore\ 54 = a \times 27$$

$$\therefore\ a = \frac{54}{27}$$

$$= 2$$

The first term of the sequence is 2.

Example 2

Find the common ratio of the geometric progression with first term 5 and 7^{th} term 320, and list the first five terms.

$a = 5,\ n = 7, T_7 = 320$

$$T_n = ar^{(n-1)}$$
$$T_7 = 5 \times (r)^{7-1}$$
$$\therefore\ 320 = 5 \times (r)^6$$
$$\therefore\quad r^6 = \frac{320}{5}$$
$$= 64$$
$$= (+2)^6 \text{ or } (-2)^6$$
$$\therefore\quad r = 2 \text{ or } -2$$

There are two values for the common ratio. Hence, there are two geometric progressions satisfying the given conditions.

First five terms of the progression with $r = 2$ are 5, 10, 20, 40, 80.
First five terms of the progression with $r = -2$ are 5, – 10, 20, – 40, 80.

Example 3

Which term is $\frac{1}{64}$, of the geometric progression with first term 64 and common ratio $\frac{1}{4}$?

$a = 64,\ r = \frac{1}{4},\ T_n = \frac{1}{64}$

$$T_n = ar^{n-1}$$
$$\frac{1}{64} = 64 \times \left(\frac{1}{4}\right)^{(n-1)}$$
$$\left(\frac{1}{4}\right)^{(n-1)} = \frac{1}{64 \times 64}$$
$$\left(\frac{1}{4}\right)^{(n-1)} = \frac{1}{4^6}$$
$$\left(\frac{1}{4}\right)^{(n-1)} = \left(\frac{1}{4}\right)^6$$
$$(n-1) = 6$$
$$n = 6 + 1$$
$$= 7 \quad \therefore \text{ it is the 7th term that is equal to } \frac{1}{64}.$$

Example 4

The first term of a geometric progression, is 160 and the 6th term is 1215. Find the common ratio of the progression.

$a = 160, \ T_6 = 1215, n = 6$

$$T_n = ar^{(n-1)}$$

$$1215 = 160\,(r)^{6-1}$$
$$160r^5 = 1215$$
$$\therefore \quad r^5 = \frac{1215}{160}$$
$$= \frac{243}{32}$$
$$= \frac{3^5}{2^5}$$
$$= \left(\frac{3}{2}\right)^5$$
$$\therefore \ r = \frac{3}{2}$$
$$= 1\frac{1}{2}$$

$\therefore$ The common ratio is $1\frac{1}{2}$.

Similarly, when any two terms of a geometric progression are given, $T_n = ar^{n-1}$ can be used to find the first term and the common ratio. Consider the following example.

Example 5

Find the common ratio and the first term of the geometric progression with 3rd term 48 and 6th term 3072.

Let us first construct two equations from the given information.

$$T_n = ar^{n-1}$$
$$T_3 = ar^{(3-1)}$$
$$ar^2 = 48 \text{ ——— } ①$$
$$T_6 = ar^{(6-1)}$$
$$ar^5 = 3072 \text{ ——— } ②$$

Both unknowns, a and r, appear in both equations 1 and 2. We can easily remove a from these two by dividing the equations.

② ÷ ①
$$\frac{ar^5}{ar^2} = \frac{3072}{48}$$
$$r^3 = 64$$
$$r^3 = 4^3$$
$$r = 4$$

Substitute $r = 4$ to ①
$$ar^2 = 48$$
$$a\,(4)^2 = 48$$
$$16a = 48$$
$$a = \frac{48}{16}$$
$$a = 3$$

First term of the progression $= 3$

Common ratio $= 4$

Example 6

The 6th term of a geometric progression is – 8 and the 10th term of the same progression is – 128.

(i) Show that there are two such geometric progressions.

(ii) Write down the first five term of each progression.

(i) $T_n = ar^{(n-1)}$

$T_6 = ar^{(6-1)}$

$ar^5 = -8$ ——— ①

$T_{10} = ar^{(10-1)}$

$ar^9 = -128$ ——— ②

② ÷ ①
$$\frac{ar^9}{ar^5} = \frac{-128}{-8}$$
$$r^4 = 16$$
$$r^4 = 2^4 \text{ or } (-2)^4$$
$$r = 2 \text{ or } -2$$

Because there are two values for the common ratio, there are two such progressions.

(ii) Substituting $r = 2$, to ①
$$ar^5 = -8$$
$$a\,(2)^5 = -8$$
$$a \times 32 = -8$$

$$a = \frac{-8}{32}$$
$$a = -\frac{1}{4}$$

First five terms of the geometric progression with $r = 2$ and $a = -\frac{1}{4}$ are $-\frac{1}{4}, -\frac{1}{2}, -1, -2, -4$.

Substituting $r = -2$, to ①

$$ar^5 = -8$$
$$a(-2)^5 = -8$$
$$a \times (-32) = -8$$
$$a = \frac{-8}{-32}$$
$$a = \frac{1}{4}$$

First five terms of the geometric progression with $r = -2$ and $a = \frac{1}{4}$ are $\frac{1}{4}, -\frac{1}{2}, 1, -2, 4$.

Exercise 16.3

1. Find the first term of the geometric progression with common ratio 3 and 4^{th} term 108.

2. Find the first term of the geometric progression with 6^{th} term 1701 and common ratio 3.

3. Find the first term of the geometric progression with common ratio $\frac{1}{2}$ and 8^{th} term 96.

4. The first term of a geometric progression is 5. The 4^{th} term is 135. Find the common ratio of the progression.

5. The common ratio of a geometric progression is 2 and its first term is 7. Which term is equal to 448?

6. The common ratio of a geometric progression is 2 and the first term is $\frac{1}{32}$. Which term is equal to 256?

7. Which term is equal to $3\frac{5}{9}$ of the geometric progression with first term 27 and common ratio $\frac{2}{3}$?

8. Write down the first five terms of the geometric progression with first term 8 and 6^{th} term -256.

9. Show that there are two geometric progressions with first term 64 and ninth term $\frac{1}{4}$, and write the first three terms of each such progression.

10. If the 4th term of a geometric progression is 48 and the 7th term is 384, then find the common ratio and the first term of the progression.

11. Show that there are two geometric progressions with 3rd term – 45 and 5th term – 1125.

12. The 4th term of a geometric progression is 100 and the 9th term is $3\frac{1}{8}$. Write the first five terms of the progression.

13. Show that there are two geometric progressions with fifth term as 40 and 9th term as 640 and write the first five terms of each of the progressions.

16.4 The sum of the first *n* terms of a geometric progression

The sum of the first n terms of a geometric progression with first term a and common ratio r is denoted by S_n. Let us now consider finding a formula for S_n.

We can write the first n terms of the geometric progression as,

$$T_1 = a,\ T_2 = ar,\ T_3 = ar^2,\ T_4 = ar^3,\ ....,\ T_n = ar^{(n-1)}.$$

$$S_n = T_1 + T_2 + T_3 + T_4 + ... + T_n$$

$$\therefore\ S_n = a + ar + ar^2 + ar^3 + ... + ar^{(n-1)} \text{———} ①.$$

We use the following technique to find a formula for S_n. First we will multiply both sides of equation ① by r. Then we get,

$$r\,S_n = ar + ar^2 + ar^3 + ar^4 + ... + ar^n \text{———} ②.$$

Now, when we subtract ① from ②, we get,

$r\,S_n - S_n = ar^n - a$ (observe that many terms on the right hand side cancel)

$$\therefore\ S_n\,(r-1) = a\,(r^n-1)$$

$$\therefore\ S_n = \frac{a\,(r^n-1)}{(r-1)} \quad (r \neq 1)$$

This is an expression for S_n in terms of a, r, n. By multiplying both the numerator and the denominator of the right hand side by – 1, we can also express it as

$$S_n = \frac{a\,(1-r^n)}{(1-r)}$$

To find S_n, one can use either $S_n = \dfrac{a\,(r^n - 1)}{(r - 1)}$ or $S_n = \dfrac{a\,(1 - r^n)}{(1 - r)}$ appropriately.

Example 1

Find the sum of the first five terms of the geometric progression 2, 6, 18, ..., both by finding the first five terms and adding them up, and by using the formula for S_n .

Let us first find the sum by adding all five terms. We are given that

$T_1 = 2,\ T_2 = 6$ and $T_3 = 18$

Moreover,

$T_4 = 18 \times 3 = 54$ and

$T_5 = 54 \times 3 = 162$.

Therefore, $S_5 = T_1 + T_2 + T_3 + T_4 + T_5$

$$= 2 + 6 + 18 + 54 + 162$$
$$= 242$$

Now let us use $S_n = \dfrac{a\,(r^n - 1)}{(r - 1)}$ to find the sum.

Because $a = 2,\ r = \dfrac{6}{2} = 3,\ n = 5.$

and $S_n = \dfrac{a\,(r^n - 1)}{r - 1}$

$$S_5 = \frac{2\,(3^5 - 1)}{3 - 1}$$

$$= \frac{2\,(243 - 1)}{2}$$

$$= \frac{2 \times 242}{2}$$

$$= 242$$

Sum of the first five terms is 242.

When the values of the terms are large or when there are many terms to add, it is easier to to use the formula for S_n.

Example 2

Find the sum of the first 6 terms of the geometric progression 120, – 60, 30, ... using the formula.

$$a = 120,\ r = \frac{-60}{120} = -\frac{1}{2},\ n = 6$$

Substituting into $S_n = \dfrac{a\,(1 - r^n)}{1 - r}$ gives us

$$S_6 = \frac{120\left[1 - \left(-\frac{1}{2}\right)^6\right]}{1 - \left(-\frac{1}{2}\right)}$$

$$= \frac{120\left[1 - \left(\frac{1}{64}\right)\right]}{\left(\frac{3}{2}\right)}$$

$$= \left[120 \times \frac{63}{64}\right] \div \frac{3}{2}$$

$$= \left[120 \times \frac{63}{64}\right] \times \frac{2}{3}$$

$$= \frac{315}{4}$$

$$= 78\,\frac{3}{4}$$

Therefore, the sum of the first six terms is $78\,\frac{3}{4}$.

There are four unknowns in the formula $S_n = \dfrac{a\,(1 - r^n)}{1 - r}$. These are a, r, n and S_n. When any three of these are given, we can find the value of the remaining unknown by using this formula. Let us consider some examples of this type.

Example 3

Find how many terms of the initial terms need to be added from the geometric progression 5, 15, 45, ... in order for the sum to be equal to 1820.

$$a = 5,\ r = \frac{15}{5} = 3,\ S_n = 1820$$

$$S_n = \frac{a\,(r^n - 1)}{r - 1}$$

$$1820 = \frac{5\,(3^n - 1)}{3 - 1}$$

$$1820 = \frac{5\,(3^n - 1)}{2}$$

$$2 \times 1820 = 5\,(3^n - 1)$$

$$\frac{2 \times 1820}{5} = 3^n - 1$$

$$728 = 3^n - 1$$

$$1 + 728 = 3^n$$

$$729 = 3^n$$

$$3^6 = 3^n$$

$$n = 6$$

First six terms need to be added.

Exercise16.4

1. Find the sum of the first 5 terms of the geometric progression with first term 4 and common ratio 3, both by writing the first 5 terms and adding them up and also using the formula.

2. Find the sum of the first 5 terms of the geometric progression 2, 8, 32, ...

3. Find the sum of the first 6 terms of the geometric progression with first term 72 and common ratio $\frac{1}{3}$.

4. Find the sum of the first 7 terms of the geometric progression 3, – 6, 12, ...

5. Find the sum of the first 6 terms of the geometric progression 18, 12, 8, ...

6. Show that the sum of the first 6 terms of the geometric progression 18, 6, 2, ... is $26\frac{26}{27}$.

7. How many of the initial terms should be added from the geometric progression 2, 4, 8, ... for the sum to be equal to 2046.

8. How many of the initial terms should be added from the geometric progression with first term 4 and common ratio 2 for the sum to be equal to 1020.

9. Find the number of initial terms that need to be added from the geometric progression 3, – 12, 48, **...** for the sum to be equal to 9831.

16.5 Problem solving related to geometric progressions

In this section we will discuss various types of problems related to geometric progressions, that we have not discussed above.

Example 1

Of a geometric progression, the sum of the first and second terms is equal to 9 and the sum of the 4th and 5th terms is equal to – 72. Find the first 5 terms of the progression.

$T_1 = a, \ T_2 = ar$

$$a + ar = 9$$
$$a(1 + r) = 9 \quad \text{——— ①}$$

$T_4 = ar^3, \ T_5 = ar^4$

$$ar^3 + ar^4 = -72$$
$$ar^3(1 + r) = -72 \quad \text{——— ②}$$

② ÷ ① $$\frac{ar^3(1 + r)}{a(1 + r)} = \frac{-72}{9}$$
$$r^3 = -8$$
$$r^3 = (-2)^3$$
$$r = -2$$

Substituting $r = -2$, to ①

$$a[1 + (-2)] = 9$$
$$a \times (-1) = 9$$
$$a = -9$$

First five terms of the progression are

$-9, 18, -36, 72, -144.$

Example 2

The first three terms of a geometric progression are respectively $(x + 2)$, $(x + 12)$, $(x + 42)$. Find the first term and the common ratio of the progression.

$$r = \frac{x + 12}{x + 2} = \frac{x + 42}{x + 12}$$

$$\frac{x + 12}{x + 2} = \frac{x + 42}{x + 12}$$

$(x + 12)(x + 12) = (x + 2)(x + 42)$

$$x^2 + 24x + 144 = x^2 + 44x + 84$$
$$144 - 84 = 20x$$
$$60 = 20x$$
$$x = \frac{60}{20}$$
$$x = 3$$

First 3 terms of the sequence are

$(3 + 2), (3 + 12), (3 + 42)$

$5, 15, 45$

First term of the sequence $= 5$

Common ratio of the sequence $= \frac{15}{5}$

$= 3$

Exercise 16.5

1. The sum of the second and third terms of a geometric progression is 21. Sum of the fifth and sixth terms is 168. Find the first 5 terms of the progression.

2. The first three terms of a geometric progression are respectively $4, (x + 3)$ and $(x + 27)$.

(i) Find the value of x.

(ii) Show that there are two such geometric progressions and find the first 4 terms of each progression.

3. The sum of the first n terms of a progression is given by $4\ (3^n - 1)$.

(i) Show that this sequence is a geometric progression.

(ii) Find the first 4 terms.

4. The first three terms of a geometric progression are such that they are the first, third and sixth terms of an arithmetic progression. The fifth term of the arithmetic progression is 15. Find the first 4 terms of the geometric progression.

5. The n^{th} term of a progression is $3(2)^{n+1}$.

(i) Show that this is a geometric progression.

(ii) Find the first term and the common ratio of the progression.

6. The first term of a geometric progression is 9. The sum of the first three terms of the progression is 7.

(i) Show that there are two such geometric progressions.

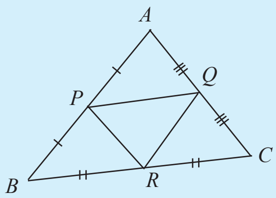
(ii) Write the first four terms of each progression.

Review Exercise – Term 2

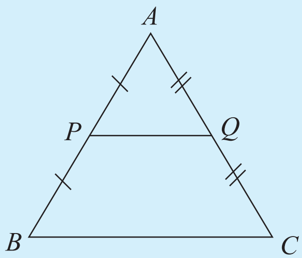
Part 1

1. For the collection of numbers 5, 3, 7, 13, 11, 9, 7, 10, 2, 3, 7 write down the following.

   (i) The mode (ii) The median (iii) The mean (iv) The inter quartile range

2. 

   If the perimeter of the triangle ABC is 24 cm, what is the perimeter of the triangle PQR?

3. 

   In the triangle ABC the midpoints of the sides AB and AC are P and Q respectively. If the perimeter of triangle APQ is 21 cm, what is the perimeter of triangle ABC?

4. A businessman who invests in the stock market bought shares in a certain company when the market price was Rs 50 per share. Later he sold all the shares when the market price was Rs 58 per share. Find the capital gain percentage of the investment the businessman made.

5. An item could be bought for Rs 15 000 by paying cash. Kavindu purchased this item on an installment basis, where the installments were calculated on the reducing balance. She made an initial payment of Rs 3000 and paid off the balance in ten equal monthly installments of Rs 1464. Find the total amount that Kavindu paid for the item.

6. $x = 2$ is one root of the equation $x^2 - ax + 18 = 10$.

   (i) Find the value of a.

   (ii) Find the other root of the equation.

7. Find the solutions of the equation $(x-2)^2 = x-2$.

8. Solve $3x^2 - 27 = 0$.

9. The sum of the squares of two successive positive integers is 145. Find the two numbers.

10. Determine the following without drawing the graph of the function $y = x^2 + 6x + 5$.
    (i) The axis of symmetry.
    (ii) The minimum value of the function.

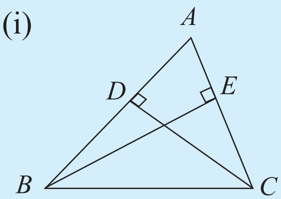
11. Write down the x coordinates of the points of intersection of the graph of the function $y = (x-2)(x+1)$ and the x - axis.

12. If $\frac{2}{x} + \frac{1}{y} = \frac{5}{6}$ and $\frac{2}{x} - \frac{1}{y} = \frac{1}{6}$, find the values of x and y.

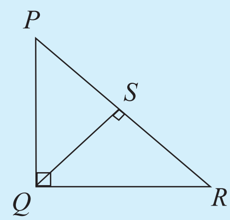
13. With reasons, determine what type of progression has $T_n = 2 \times 3^n$ as its nth term.

14. $AB = 6$ cm, $BC = 7$ cm and $AC = 4$ cm in the triangle ABC. x is a variable point on the side BC. If the midpoint of AX is P, describe the locus of P.

15. (i)



(ii)



Show that ,

the pair of triangles ABE and ADC in figure (i) are equiangular.
the pair of triangles PQS and QSR in figure (ii) are equiangular.

Part II

1. When the length of a certain rectangle is reduced by 6 units and the breadth is increased by 2 units, the area of the new rectangle formed is 12 square units less than the area of the original rectangle. By taking the length and the breadth of the initial rectangle as x and y respectively, answer the following.

    (i) Express the length and the breadth of the second rectangle in terms of x and y.

(ii) Express the area of the second rectangle in terms of x and y.
(iii) Construct an equation in terms *of* x and y.
(iv) Show that the length of the initial rectangle is three times its breadth.
(v) If the area of the original rectangle is 192 square units, find its length and breadth.

2. The third term of a geometric progression with a positive common ratio is 3 more than the second term, and the fifth term of the progression is 12 more than the fourth term.

(i) Find the common ratio and the initial term of the progression.
(ii) Write down the first five terms of the progression.
(iii) Show that the n^{th} term of the progression is $3 \times 2^{n-2}$.

3. A person who invests in the stock market bought 5000 shares in company A which pays annual dividends of Rs 1.25 per share, and a certain number of shares in company B which pays annual dividends of Rs 1.50 per share. When the market price of a share in company A and company B were Rs 30 and Rs 35 respectively, he sold all the shares he owned in the two companies and bought shares in company C at the market price of Rs 50 per share. Company C pays annual dividends of Rs 2.50 per share. His annual dividends income from the investment in company C was Rs 12 750.

(i) Find the number of shares he owned in company *B*.
(ii) Show that the investment in company C resulted in an increase of Rs 2000 in his annual dividends income.

4. A person took a loan of Rs 10 000 at an annual compound interest rate of 8% on the assurance of settling the loan in two years. However, he was unable to pay back the loan in two years as promised. He paid Rs 6000 to the money lender at the end of the second year and came to an agreement with him to pay off the remaining loan amount together with the interest by the end of the following year. This agreement was reached on the condition that he would pay a higher interest for that year.

(i) Calculate the interest for the first year.
(ii) Find the total amount that he needed to pay at the end of the second year to settle the loan.
(iii) How much remained to be paid off at the beginning of the third year?
(iv) If he settled the loan as promised at the end of the third year by paying Rs 6230.40, find the interest rate that was charged for the third year.

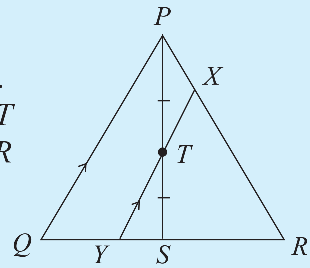
5. The straight line drawn through B, parallel to the diagonal AC of the parallelogram $ABCD$, meets the side DC produced at E. The straight lines AE and BC intersect at P, and the diagonals AC and BD intersect at Q.

(i) Draw a sketch and mark the given information.

(ii) Prove that $ABEC$ is a parallelogram.

(iii) Prove that $PQ = \frac{1}{4} DE$.

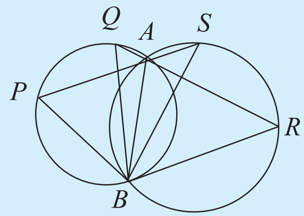
6. The midpoint of the side QR of the triangle PQR is S. The midpoint of PS is T. The line drawn through T parallel to PQ meets the side PR at X and the side QR at Y.



(i) Prove that $YT = \frac{1}{2} PQ$.

(ii) Prove that $XY = \frac{3}{4} PQ$.

7. (a) Based on the information in the figure,

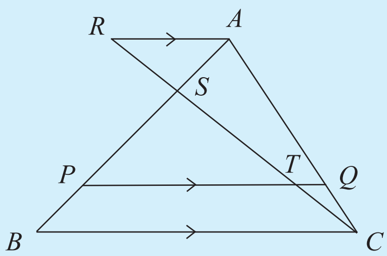


(i) name an angle equal to $A\hat{P}B$.

(ii) prove that the triangles BPS and BQR are equiangular .

(iii) prove that $BP : BQ = BS : BR$.

(b) Based on the information in the figure,



(i) prove that $\frac{PQ}{BC} = \frac{AQ}{AC}$.

(ii) prove that $\frac{PQ}{BC} = \frac{RT}{RC}$.

8. (i) Prepare a table of values to draw the graph of the function $y = x\,(x - 2)$ in the interval $-3 \leq x \leq 5$.

(ii) Select a suitable scale along the x and y axes and draw the graph of $y = x\,(x - 2)$.

(iii) By considering the graph,

(a) write down the axis of symmetry of the graph.

(b) the minimum value of the function.

(c) the values of x for which the value of the function is 0.

(d) the roots of the equation $x(x-2)=0$.

(e) the range of values of x for which the function is negative.

(iv) Draw the graph $y=x^2$ and find the value of $\sqrt{2}$ to the first decimal place, using the graph.

ලඝුගණක

மடக்கைகள்

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences 1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

ලඝුගණක

மடக்கைகள்

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences 1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Glossary

A

Axis of symmetry	සමමිති අක්ෂය

C

Class Boundaries	පන්ති මායිම්
Class intervals	පන්ති ප්‍රාන්තර
Class Limits	පන්ති සීමා
Class width	පන්තියක තරම
Coefficient	සංගුණකය
Common Ratio	පොදු අනුපාතය
Completing the Square	වර්ග පූර්ණය
Compound Interest	වැල් පොලිය
Continuous data	සන්තතික දත්ත
Converse	විලෝමය
Cumulative Frequency	සමුච්චිත සංඛ්‍යාතය
Cumulative Frequency	සමුච්චිත සංඛ්‍යාතය වක්‍රය

D

Data	දත්ත
Discrete data	විවික්ත දත්ත
Domain	වසම

F

First Term	පළමුවන පදය
Frequency	සංඛ්‍යාතය
Frequency polygon	සංඛ්‍යාත බහුඅස්‍රය
Function	ශ්‍රිතය

G

Geometric progression	ගුණෝත්තර ශ්‍රේඪී

H

Histogram	ජාල රේඛය

I

Instalment	වාරිකය

M

Maximum value	උපරිම අගය
Mid point	මධ්‍ය ලක්ෂ්‍ය
Minimum value	අවම අගය

N

Number of month units	මාස ඒකක ගණන
Number Sequence	සංඛ්‍යා අනුක්‍රම

P

Preceding Term	පෙර පදය
Proof	සාධනය
Proportional	සමානුපාතික

Q

Quadratic Equation	වර්ගජ සමීකරණ

R

Range	පරාසය / ප්‍රාන්තරය
Reducing Balance	හීනවන ශේෂය
Rider	අනුමේය

S

Simultaneous equations	සමගාමී සමීකරණ
Solutions	විසඳුම
Successive Term	පසු පදය

T

Turning point	හැරුම් ලක්ෂ්‍යය

U

Unknown	අඥාතය

V

Verification	සත්‍යාපනය

Sequence of the Lessons

Chapter of Textbook	No.of Periods
1 Term	
1. Real Numbers	10
2. Indices and Logarithms I	08
3. Indices and Logarithms II	06
4. Surface Area of Solids	05
5. Volume of the Solids	05
6. Binomial Expressions	04
7. Algebraic Fractions	04
8. Areas of Plane Figures between Parallel Lines	12
2 Term	
09. Percentages	06
10. Share Market	05
11. Mid Point Theorem	05
12. Graphs	12
13. Formulae	10
14. Equiangular Triangles	12
15. Data representaion and Interpretation	12
16. Geometric Progressions	06
3 Term	
17. Pythagoras's Theorem	04
18. Trignometry	12
19. Matrices	08
20. Inequalities	06
21. Cyclic Quadrilaterals	10
22. Tangent	10
23. Constructions	05
24. Sets	06
25. Probability	07

MATHEMATICS

Grade 11

Part - III

Educational Publications Department

To obtain electronic textbooks, visit
www.edupub.gov.lk

First Print 2015
Second Print 2016
Third Print 2017
Fourth Print 2018
Fifth Print 2019

All Rights Reserved

ISBN 978-955-25-0311-5

Published by Educational Publications Department
Printed by State Printing Corporation

The National Anthem of Sri Lanka

Sri Lanka Matha
Apa Sri Lanka Namo Namo Namo Namo Matha
Sundara siri barinee, surendi athi sobamana Lanka
Dhanya dhanaya neka mal palaturu piri jaya bhoomiya ramya
Apa hata sepa siri setha sadana jeewanaye matha
Piliganu mena apa bhakthi pooja Namo Namo Matha
Apa Sri Lanka Namo Namo Namo Namo Matha
Oba we apa vidya
Obamaya apa sathya
Oba we apa shakthi
Apa hada thula bhakthi
Oba apa aloke
Apage anuprane
Oba apa jeevana we
Apa mukthiya oba we
Nava jeevana demine, nithina apa pubudukaran matha
Gnana veerya vadawamina regena yanu mana jaya bhoomi kara
Eka mavakage daru kela bevina
Yamu yamu vee nopama
Prema vada sema bheda durerada
Namo, Namo Matha
Apa Sri Lanka Namo Namo Namo Namo Matha

අපි වෙමු එක මවකගෙ දරුවෝ
එක නිවසෙහි වෙසෙනා
එක පාටැති එක රුධිරය වේ
අප කය තුළ දුවනා

එබැවිනි අපි වෙමු සොයුරු සොයුරියෝ
එක ලෙස එහි වැඩෙනා
ජීවත් වන අප මෙම නිවසේ
සොඳින සිටිය යුතු වේ

සැමට ම මෙත් කරුණා ගුණෙනී
වෙළී සමගි දමිනී
රන් මිණි මුතු නො ව එය ම ය සැපතා
කිසි කල නොම දිරනා

ආනන්ද සමරකෝන්

ஒரு தாய் மக்கள் நாமாவோம்
ஒன்றே நாம் வாழும் இல்லம்
நன்றே உடலில் ஓடும்
ஒன்றே நம் குருதி நிறம்

அதனால் சகோதரர் நாமாவோம்
ஒன்றாய் வாழும் வளரும் நாம்
நன்றாய் இவ் இல்லினிலே
நலமே வாழ்தல் வேண்டுமன்றோ

யாவரும் அன்பு கருணையுடன்
ஒற்றுமை சிறக்க வாழ்ந்திடுதல்
பொன்னும் மணியும் முத்துமல்ல - அதுவே
யான்று மழியாச் செல்வமன்றோ.

ஆனந்த சமரக்கோன்
கவிதையின் பெயர்ப்பு.

Being innovative, changing with right knowledge
Be a light to the country as well as to the world.

Message from the Hon. Minister of Education

The past two decades have been significant in the world history due to changes that took place in technology. The present students face a lot of new challenges along with the rapid development of Information Technology, communication and other related fields. The manner of career opportunities are liable to change specifically in the near future. In such an environment, with a new technological and intellectual society, thousands of innovative career opportunities would be created. To win those challenges, it is the responsibility of the Sri Lankan government and myself, as the Minister of Education, to empower you all.

This book is a product of free education. Your aim must be to use this book properly and acquire the necessary knowledge out of it. The government in turn is able to provide free textbooks to you, as a result of the commitment and labour of your parents and elders.

Since we have understood that the education is crucial in deciding the future of a country, the government has taken steps to change curriculum to suit the rapid changes of the technological world. Hence, you have to dedicate yourselves to become productive citizens. I believe that the knowledge this book provides will suffice your aim.

It is your duty to give a proper value to the money spent by the government on your education. Also you should understand that education determines your future. Make sure that you reach the optimum social stratum through education.

I congratulate you to enjoy the benefits of free education and bloom as an honoured citizen who takes the name of Sri Lanka to the world.

Akila Viraj Kariyawasam
Minister of Education

Foreword

The educational objectives of the contemporary world are becoming more complex along with the economic, social, cultural and technological development. The learning and teaching process too is changing in relation to human experiences, technological differences, research and new indices. Therefore, it is required to produce the textbook by including subject related information according to the objectives in the syllabus in order to maintain the teaching process by organizing learning experiences that suit to the learner needs. The textbook is not merely a learning tool for the learner. It is a blessing that contributes to obtain a higher education along with a development of conduct and attitudes, to develop values and to obtain learning experiences.

The government in its realization of the concept of free education has offered you all the textbooks from grades 1-11. I would like to remind you that you should make the maximum use of these textbooks and protect them well. I sincerely hope that this textbook would assist you to obtain the expertise to become a virtuous citizen with a complete personality who would be a valuable asset to the country.

I would like to bestow my sincere thanks on the members of the editorial and writer boards as well as on the staff of the Educational Publications Department who have strived to offer this textbook to you.

W. M. Jayantha Wickramanayaka,
Commissioner General of Educational Publications,
Educational Publications Department,
Isurupaya,
Battaramulla.
10.04.2019

Monitoring and Supervision

W. M. Jayantha Wickramanayaka - Commissioner General of Educational Publications
Educational Publications Department

Direction

W.A. Nirmala Piyaseeli - Commissioner (Development)
Educational Publications Department

Co-ordination

Thanuja Maithrie Witharana - Assistant Commissioner
Educational Publications Department

Board of Editors

Dr. Romaine Jayewardene - Senior Lecturer, University of Colombo
Dr. D.K. Mallawa Arachchi - Senior Lecturer, University of Kelaniya
Dr. Sri Dharan - Senior Lecturer, University of Colombo
Dr. Nalin Ganegoda - Senior Lecturer, University of Sri Jayawardenepura
B.D.C. Biyanwila - Director of Education, Ministry of Education
G.P.H. Jagath Kumara - Senior Lecturer National Institute of Education
Thanuja Maithrie Witharana - Assistant Commissioner
Educational Publications Department

Board of Writers

Dr. Rochana Meegaskumbura - Senior Lecturer, University of Peradeniya
Dr. J. K. Rathnayake - Senior Lecturer, University of Colombo
Dr. Jayantha Senadeera - Senior Lecturer, Sri Lanka Open University
H.M.A. Jayasena - In Service Adviser (Retired)
Y.V.R. Witharana - In Service Adviser, Zonal Education Office, Dehiowita
W.M.W.C. Walisinghe - In Service Adviser, Zonal Education Office, Kegalle
Ajith Ranasinghe - In Service Adviser, Zonal Education Office, Homagama
Anura Weerasinghe - In Service Adviser, (Priven), Matara District
W.M.D. Lal Wijekantha - Teacher, St. Thomas' College, Mt. Lavinia
N. Vakeeshamoorthi - Director of Education (Retired)
R.S.E. Pushparajan - Assistant Director, Zonal Education Office, Puttalam
V. Murali - Teacher Educator Service Zonal Education Office, Vavuniya

Language Editing

M.A.K. Samurdhi - Teacher, Sirimavo Bandaranaike Vidyalaya, Colombo 07

Proof Reading

Edward R. Peiris - Engineer (Retired)

Illustrations, Cover Page and Technical Assistance

B.T. Chathurani Perera
R.D. Thilini Sewwandi
- Computer Assistant, Educational Publications Department

Message of the Board of Compilers

This textbook has been compiled in accordance with the new syllabus to be implemented from 2015.

Textbooks are compiled for students. Therefore, we have made an attempt to compile this textbook in a simple and detailed manner making it possible for you to read and understand it on your own.

We have included descriptions, activities and examples to introduce the subject concepts in an attractive manner and to establish them. Moreover, activities are organized from simple to complex to develop an interest to do them.

We have used the terms related to mathematical concepts in accordance with the glossary of technical terms of mathematics compiled by the Department of Official Languages.

Some subject matter learnt during the earlier grades is necessary to learn the subject content in the grade 11 syllabus. Thus, review exercises are included at the beginning of each chapter to revise previous knowledge. You will be prepared by them for the subject content of grade 11.

In addition, students may use the grade 10 book which you have if you need to recall previous knowledge.

You will gain maximum benefit from this textbook by reading the chapters and doing the review exercises of each chapter even before your teacher teaches them in the classroom.

We hope that studying mathematics will be an interesting, joyful and productive experience.

Board of Compilers

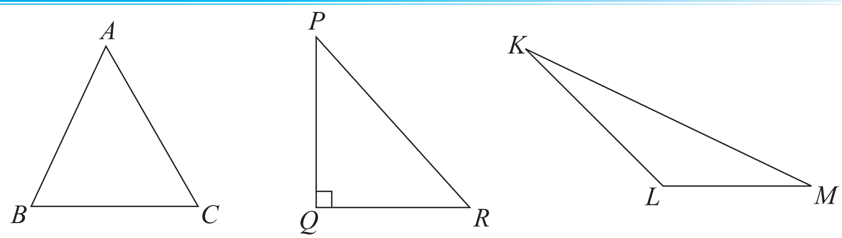
Contents

17 Pythagoras' Theorem

By studying this lesson you will be able to

- identify Pythagoras' Theorem
- use Pythagoras' Theorem in calculations and to prove riders
- identify Pythagorean triples.

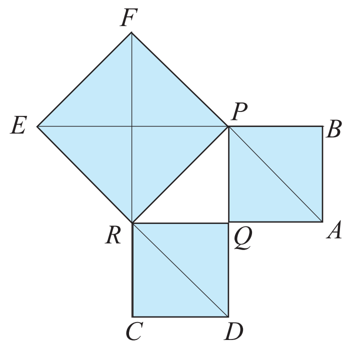
Introduction



The triangles ABC, PQR and KLM in the above figure are respectively an acute angled triangle, a right angled triangle and an obtuse angled triangle. These triangles have been thus named, by considering the largest (one or more) of the interior angles. Accordingly, the right angle $P\hat{Q}R$ is the largest interior angle of the triangle PQR. The side PR which is directly opposite this angle is the longest side of the triangle. This side is called the hypotenuse, and the remaining sides, namely PQ and QR, are known as the sides that include the right angle.

There is evidence to show that from ancient times, man knew about the geometrical properties of triangles. The marvel of the pyramids made in Egypt around 3000 B.C. is accepted by us all. For such creations, knowledge of geometry, especially on the characteristics of triangles is essential. In the "Rhind Papyrus" of around 1650 B.C. too, the main shape that can be observed is the triangle. Using this knowledge on the geometry of triangles, in 600 B.C., the Greek mathematician Pythagoras presented a special relationship between the lengths of the sides of right angled triangles. Although there is evidence to show that this relationship was known by early civilizations in countries such as China and India, Pythagoras is considered to be the first to offer a geometrical proof of this relationship. Later on in 300 B.C., the mathematician named Euclid included this result as a theorem, together with its proof, in his historical book called **THE ELEMENTS**.

17.1 Pythagoras' Theorem



A part of a floor on which tiles of the same shape and size have been placed is depicted in the above figure. The shape of each tile is an isosceles right angled triangle. Let us consider the isosceles right angled triangle *PQR*. The square *PQAB* has been drawn on the side *PQ* and the square *RCDQ* has been drawn on the side *RQ* (regions shaded in blue) of this triangle. The square drawn on the side *PQ* has an area equal to that covered by two tiles. Similarly, the square drawn on the side *QR* also has an area equal to that covered by two tiles, while the square *PREF* drawn on the hypotenuse *PR* has an area equal to that covered by four tiles.

Accordingly, for the squares lying on the three sides of the isosceles right angled triangle *PQR*, it is clear that the following relationship holds.

$$\text{Area of square } PQAB + \text{Area of square } RCDQ = \text{Area of square } PREF$$

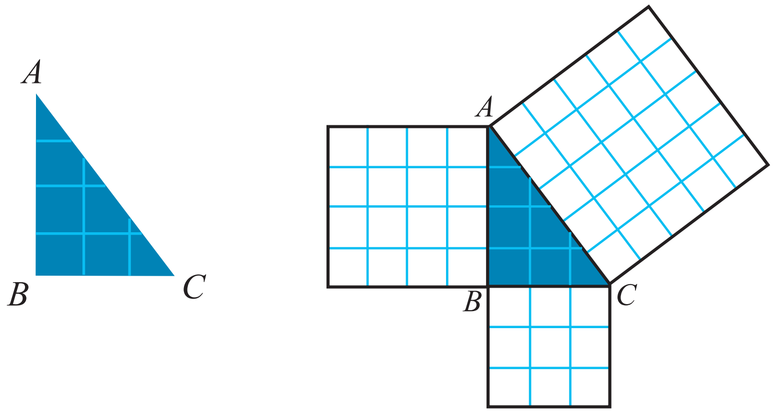
This relationship can be established further by doing the activity given below.

Activity

Using a square ruled paper, cut out 3 square shaped laminas and 1 triangular shaped lamina as follows.

(i) A square shaped lamina with side length equal to the length of 3 small squares
(ii) A square shaped lamina with side length equal to the length of 4 small squares
(iii) A square shaped lamina with side length equal to the length of 5 small squares
(iv) A right triangular shaped lamina, where the sides which include the right angle are of lengths equal to the length of 3 small squares and 4 small squares respectively.

Paste the right triangular shaped lamina on a white paper. Paste the squares on the three sides of the triangle as shown in the figure given below.



The area of the square on the side AB of the right angled triangle ABC $\Big\}$ = 16 small squares

The area of the square on the side BC = 9 small squares

The area of the square on the side AC = 25 small squares

Accordingly, the sum of the areas of the squares on the sides which include the right angle of the triangle ABC $\Big\}$ = $16 + 9$ small squares

= 25 small squares

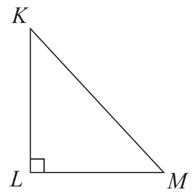
The area of the square on the hypotenuse AC of the right angled triangle ABC $\Big\}$ = 25 small squares

Therefore, in the right angled triangle ABC, the sum of the areas of the squares on the sides which include the right angle is equal to the area of the square on the hypotenuse of the triangle.

This relationship between the sides of a right angled triangle, which was known from ancient days, can be expressed as a theorem as follows.

Pythagoras' Theorem: In a right angled triangle, the area of the square drawn on the hypotenuse is equal to the sum of the areas of the squares drawn on the remaining sides of the triangle, which include the right angle.

In the right angled triangle *KLM* shown in the figure, *KM* is the hypotenuse while *KL* and *LM* are the remaining sides which include the right angle.



Area of the square drawn on the side $KL = KL^2$

Area of the square drawn on the side $LM = LM^2$

Area of the square drawn on the hypotenuse $KM = KM^2$

Therefore, according to Pythagoras' Theorem,

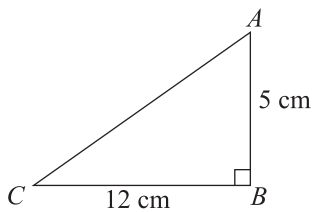
$$\boxed{KL^2 + LM^2 = KM^2}$$

Let us now consider how calculations are performed using Pythagoras' Theorem.

Example 1

In the right angled triangle *ABC*, $\hat{B} = 90^\circ$, $AB = 5$ cm and $BC = 12$ cm. Find the length of the side *AC*.

According to Pythagoras' Theorem,



$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= 5^2 + 12^2 \\ &= 25 + 144 \\ &= 169 \end{aligned}$$

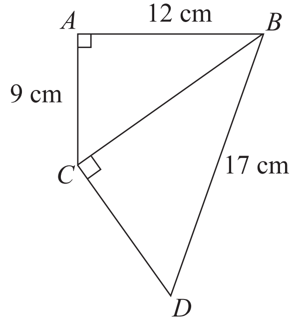
$$\therefore \quad AC = \sqrt{169} = 13$$

$\therefore$ the length of side *AC* is 13cm.

Example 2

Find the length of *CD* based on the information given in the figure.

According to the figure, Pythagoras' Theorem can be applied to the right angled triangle *ABC*.



$$\therefore\ BC^2 = AB^2 + AC^2$$
$$= 12^2 + 9^2$$
$$= 144 + 81$$
$$= 225$$
$$\therefore\ BC = \sqrt{225}$$
$$= 15$$

Applying Pythagoras' Theorem to the right angled triangle *BCD*,

$$CD^2 + BC^2 = BD^2$$
$$CD^2 + 15^2 = 17^2$$
$$CD^2 + 225 = 289$$
$$\therefore\ CD^2 = 289 - 225$$
$$= 64$$
$$\therefore\ CD = 8$$

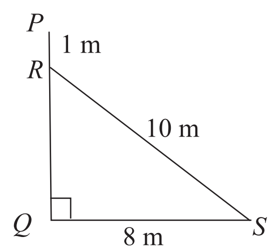
$\therefore$ the length of *CD* is 8 cm.

Now let us consider how Pythagoras' Theorem can be used to solve practical problems.

Example 3

One end of a wire is tied to a ring fastened at a point 1m below the top of a vertical utility pole, while the other end is tied to a ring fastened on the ground, 8 m away from the foot of the pole. The length of the wire between the two rings is 10 m. Find the height of the utility pole. (Assume that the wire is stretched.)

Let us draw the figure according to the given information.



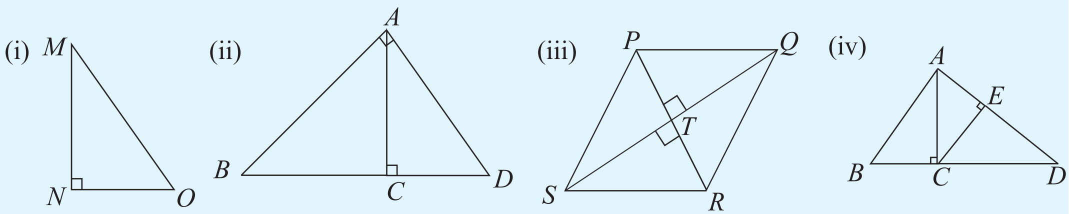
As the pole *PQ* is vertical, it makes a right angle with the horizontal ground. Therefore $P\hat{Q}S = 90^\circ$.
Since *QRS* is a right angled triangle, according to Pythagoras' Theorem,

$$QR^2 + QS^2 = RS^2$$
$$QR^2 + 8^2 = 10^2$$
$$QR^2 + 64 = 100$$
$$\therefore \quad QR^2 = 100 - 64$$
$$QR^2 = 36$$
$$\therefore \quad QR = 6$$
$$\therefore \text{ Height of the pole } = QR + PR$$
$$= 6 + 1$$
$$= 7$$
$\therefore$ Height of the pole is 7 m.

Now do the following exercise using Pythagoras' Theorem.

Exercise 17.1

1. Fill in the blanks using the information in the figure.



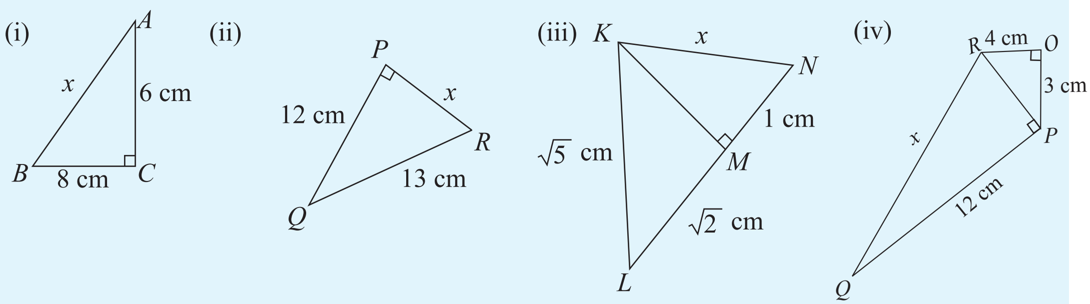
(i) $MO^2 = ... + ...$

(ii) $BD^2 = ... + ...$
$.... = AC^2 + CD^2$
$AB^2 = AC^2 + ...$

(iii) $PQ^2 = ... + ...$
$QR^2 = ... + ...$

(iv) $AB^2 = ... + AC^2$
$.... = AE^2 + EC^2$
$AD^2 = AC^2 + ...$

2. Find the value of x in each of the right angled triangles given below.



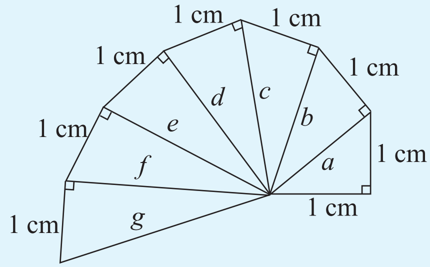
3. In the equilateral triangle ABC, D is the foot of the perpendicular drawn from the vertex A to the side BC. If the length of a side of the triangle is 2 cm, find the length of AD (Express the answer as a surd.)

4. The location Q is reached from the location P on the horizontal ground, by travelling 15m to the North from P and then 8m to the East.

(i) Draw a sketch based on the above information.
(ii) Find the distance PQ.

5. The lengths of the diagonals of a rhombus are 12 m and 16 m. Find the length of each side of the rhombus.

6. The figure illustrates the special creation Archimedes' spiral. By considering the right angled triangles in the figure, find the lengths a, b, c, d, e, f and g, using the given measurements. (Express the answers in surd form)

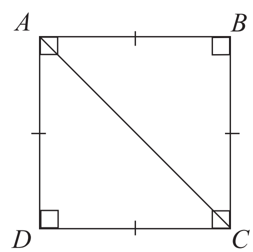


17.2 Further applications of Pythagoras' Theorem

Now let us consider how riders related to Pythagoras' Theorem are proved.

Example 1

$ABCD$ is a square. Prove that $AC^2 = 2AB^2$.



Proof : ABC is a right angled triangle since $A\hat{B}C = 90^\circ$

Applying Pythagoras' Theorem to the triangle ABC,

$AC^2 = AB^2 + BC^2$

$AC^2 = AB^2 + AB^2$ ($AB = BC$, sides of a square)

$\therefore \quad \underline{\underline{AC^2 = 2AB^2}}$

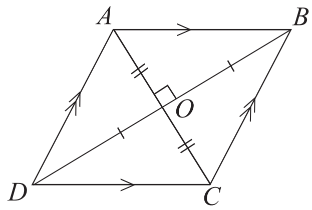
Example 2

In the rhombus $ABCD$, the diagonals AC and BD intersect at O. Prove that $AC^2 + BD^2 = 4\,AB^2$.

Proof: Since $ABCD$ is a rhombus, the diagonals bisect each other perpendicularly. (See figure)

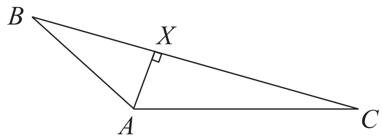
$\therefore \quad A\hat{O}B = 90^\circ$, $AO = OC$ and $BO = OD$.

According to Pythagoras' Theorem; in the right angled triangle AOB,



$$AO^2 + OB^2 = AB^2$$
$$\left(\frac{1}{2}AC\right)^2 + \left(\frac{1}{2}BD\right)^2 = AB^2$$
$$\frac{1}{4}AC^2 + \frac{1}{4}BD^2 = AB^2$$
$$\frac{1}{4}(AC^2 + BD^2) = AB^2$$
$$\therefore \quad \underline{\underline{AC^2 + BD^2 = 4\,AB^2}}$$

Example 3



In the triangle ABC, $B\hat{A}C$ is an obtuse angle.
AX is drawn from A,
perpendicular to BC. Prove that $AB^2 - AC^2 = BX^2 - CX^2$

Proof:

In the right angled triangle AXB, according to Pythagoras' Theorem

$AB^2 = AX^2 + BX^2$ —— ①

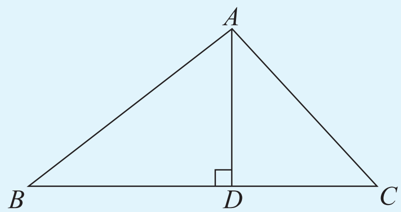
In the right angled triangle AXC, according to Pythagoras' Theorem,

$AC^2 = AX^2 + CX^2$ —— ②

① − ② ; $AB^2 - AC^2 = AX^2 + BX^2 - (AX^2 + CX^2)$
$= AX^2 + BX^2 - AX^2 - CX^2$
$= \underline{\underline{BX^2 - CX^2}}$

Prove the riders in the exercise given below as illustrated in the above examples.

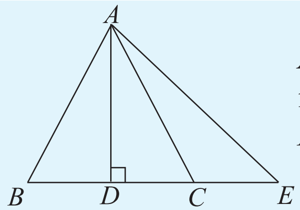
Exercise 17.2



1. AD is perpendicular to BC in the triangle ABC. (See figure)
If $AD = DC$, prove that $AB^2 = BD^2 + DC^2$.

2. AD is perpendicular to BC in the triangle ABC. Prove that $AB^2 + CD^2 = AC^2 + BD^2$.

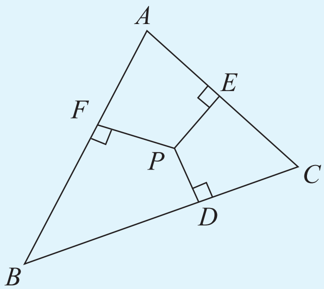
3. AD is perpendicular to BC in the equilateral triangle ABC. Prove that $4\,AD^2 = 3\,BC^2$.

4. 

AD is perpendicular to BC in the equilateral triangle ABC in the figure. BC has been produced to E such that $DC = CE$. Prove that $AE^2 = 7\,EC^2$.

5. The diagonals of the quadrilateral $ABCD$ bisect each other perpendicularly at O. Prove that $AB^2 + CD^2 = AD^2 + BC^2$.

6. O is a point within the rectangle $ABCD$. Prove that $AO^2 + CO^2 = BO^2 + DO^2$ (Hint: Draw a parallel line through O to any side of $ABCD$.)

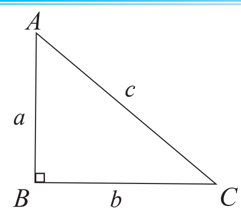
7. 

P is a point within the triangle ABC. The perpendiculars drawn from the point P to the sides BC, AC and AB meet these sides at D, E and F respectively.
Prove that,

(i) $BP^2 - PC^2 = BD^2 - DC^2$ and
(ii) $BD^2 + CE^2 + AF^2 = CD^2 + AE^2 + BF^2$

8. The two squares $ABXY$ and $BCPQ$ lie on the same side of the straight line ABC. Prove that $PX^2 + CY^2 = 3\,(AB^2 + BC^2)$

17.3 Pythagorean triples



In the right angled triangle ABC in the figure, if the lengths of the sides which include the right angle are a and b units, and the length of the hypotenuse is c units, then we know that $a^2 + b^2 = c^2$ according to Pythagoras' Theorem. Values of a, b and c which satisfy the equation

$a^2 + b^2 = c^2$ are known as Pythagorean triples.

Since $3^2 + 4^2 = 5^2$, we obtain that (3, 4, 5) is a Pythagorean triple. Any multiple of the triple (3, 4, 5) is also a Pythagorean triple.

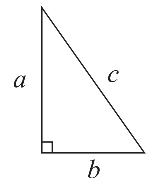
Eg: Multiplying each value of the triple (3, 4, 5) by 2 we obtain (6, 8, 10). Since $6^2 + 8^2 = 10^2$, (6, 8, 10) is a Pythagorean triple.

Multiplying each value of the triple (3, 4, 5) by 3 we obtain (9, 12, 15). Since $9^2 + 12^2 = 15^2$, (9, 12, 15) is also a Pythagorean triple.
There are Pythagorean triples apart from the multiples of (3, 4, 5).

Eg: Since $5^2 + 12^2 = 13^2$, (5, 12, 13) is a Pythagorean triple.
Since $8^2 + 15^2 = 17^2$, (8, 15, 17) is a Pythagorean triple.

A mathematician named Euclid introduced "parametric equations" to find Pythagorean triples. Given any two numbers x and y, if $a = x^2 - y^2$, $b = 2xy$ and $c = x^2 + y^2$, then $a^2 + b^2 = c^2$, and hence (a, b, c) is a Pythagorean triple.

Eg: $x = 6$, and $y = 5$, then $a = x^2 - y^2 = 6^2 - 5^2 = 11$
$b = 2xy = 2 \times 6 \times 5 = 60$
$c = x^2 + y^2 = 6^2 + 5^2 = 61$.
Therefore (11, 60, 61) is a Pythagorean triple.

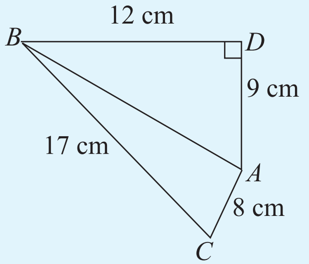


Exercise 17.3

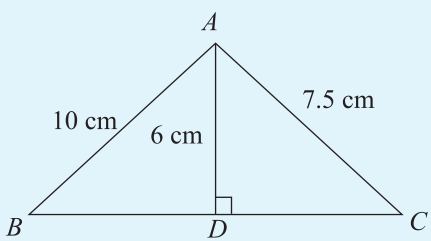
1. The following triples are the lengths of the sides of two triangles. Select the triangle which is a right angled triangle and write down the corresponding Pythagorean triple.

(i) (8, 15, 17) (ii) (14, 18, 25)

2. Based on the measurements given in figures (i) and (ii), show that $B\hat{A}C$ is a right angle in each figure.



(i)



(ii)

3. By completing the table given below, find the Pythagorean triples corresponding to the given pairs of values. Verify your answers.

x	y	x^2	y^2	a $x^2 - y^2$	b $2xy$	c $x^2 + y^2$	Pythagorean triple
2	1						
5	4						
4	3						
6	5						
7	5						

Miscellaneous Exercise

1. The chord AB of the circle with centre O, which lies at a distance of 9 cm from O, is of length 24 cm. Find the radius of the circle.

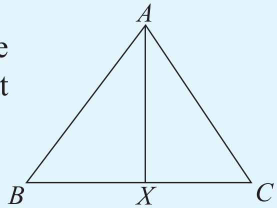
2. Construct the triangle ABC where $AB = 2$cm, $BC = 3$cm and $\hat{B}$ is a right angle. Using the triangle you constructed, find the value of $\sqrt{13}$ to the first decimal place.

3. Construct straight line segments of the lengths given below.

   (i) $\sqrt{8}$ cm (ii) $\sqrt{10}$ cm (iii) $\sqrt{41}$ cm

4. ABC is an equilateral triangle. D is the midpoint of AB and E is the midpoint of CD. Prove that $16\,AE^2 = 7\,AB^2$.

5. In the triangle ABC, $\hat{B}$ is an acute angle. The foot of the perpendicular dropped from A to BC is X. Prove that $AC^2 = AB^2 + BC^2 - 2\,BC.BX$

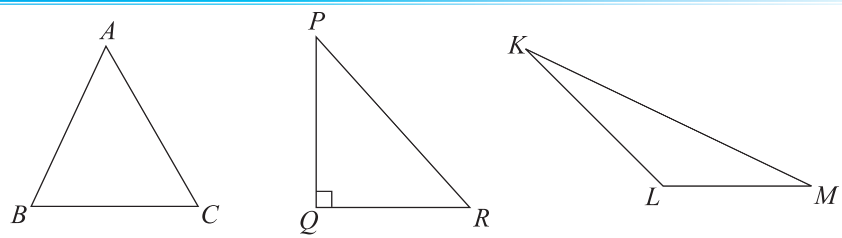


17 Pythagoras' Theorem

By studying this lesson you will be able to

- identify Pythagoras' Theorem
- use Pythagoras' Theorem in calculations and to prove riders
- identify Pythagorean triples.

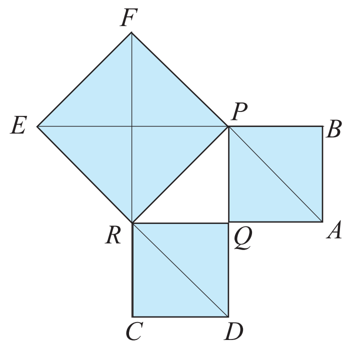
Introduction



The triangles ABC, PQR and KLM in the above figure are respectively an acute angled triangle, a right angled triangle and an obtuse angled triangle. These triangles have been thus named, by considering the largest (one or more) of the interior angles. Accordingly, the right angle $P\hat{Q}R$ is the largest interior angle of the triangle PQR. The side PR which is directly opposite this angle is the longest side of the triangle. This side is called the hypotenuse, and the remaining sides, namely PQ and QR, are known as the sides that include the right angle.

There is evidence to show that from ancient times, man knew about the geometrical properties of triangles. The marvel of the pyramids made in Egypt around 3000 B.C. is accepted by us all. For such creations, knowledge of geometry, especially on the characteristics of triangles is essential. In the "Rhind Papyrus" of around 1650 B.C. too, the main shape that can be observed is the triangle. Using this knowledge on the geometry of triangles, in 600 B.C., the Greek mathematician Pythagoras presented a special relationship between the lengths of the sides of right angled triangles. Although there is evidence to show that this relationship was known by early civilizations in countries such as China and India, Pythagoras is considered to be the first to offer a geometrical proof of this relationship. Later on in 300 B.C., the mathematician named Euclid included this result as a theorem, together with its proof, in his historical book called **THE ELEMENTS**.

17.1 Pythagoras' Theorem



A part of a floor on which tiles of the same shape and size have been placed is depicted in the above figure. The shape of each tile is an isosceles right angled triangle. Let us consider the isosceles right angled triangle PQR. The square $PQAB$ has been drawn on the side PQ and the square $RCDQ$ has been drawn on the side RQ (regions shaded in blue) of this triangle. The square drawn on the side PQ has an area equal to that covered by two tiles. Similarly, the square drawn on the side QR also has an area equal to that covered by two tiles, while the square $PREF$ drawn on the hypotenuse PR has an area equal to that covered by four tiles.

Accordingly, for the squares lying on the three sides of the isosceles right angled triangle PQR, it is clear that the following relationship holds.

Area of square $PQAB$ + Area of square $RCDQ$ = Area of square $PREF$

This relationship can be established further by doing the activity given below.

Activity

Using a square ruled paper, cut out 3 square shaped laminas and 1 triangular shaped lamina as follows.

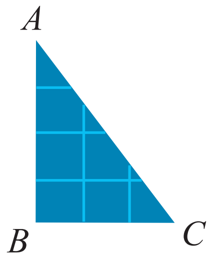
(i) A square shaped lamina with side length equal to the length of 3 small squares

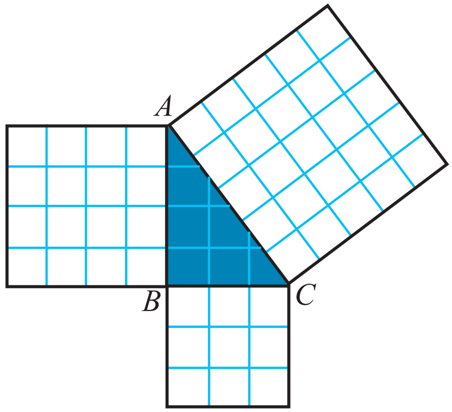
(ii) A square shaped lamina with side length equal to the length of 4 small squares

(iii) A square shaped lamina with side length equal to the length of 5 small squares

(iv) A right triangular shaped lamina, where the sides which include the right angle are of lengths equal to the length of 3 small squares and 4 small squares respectively.

Paste the right triangular shaped lamina on a white paper. Paste the squares on the three sides of the triangle as shown in the figure given below.





The area of the square on the side AB of the right angled triangle ABC = 16 small squares

The area of the square on the side BC = 9 small squares

The area of the square on the side AC = 25 small squares

Accordingly, the sum of the areas of the squares on the sides which include the right angle of the triangle ABC = $16 + 9$ small squares

= 25 small squares

The area of the square on the hypotenuse AC of the right angled triangle ABC = 25 small squares

Therefore, in the right angled triangle ABC, the sum of the areas of the squares on the sides which include the right angle is equal to the area of the square on the hypotenuse of the triangle.

This relationship between the sides of a right angled triangle, which was known from ancient days, can be expressed as a theorem as follows.

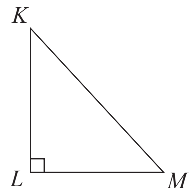
Pythagoras' Theorem: In a right angled triangle, the area of the square drawn on the hypotenuse is equal to the sum of the areas of the squares drawn on the remaining sides of the triangle, which include the right angle.

In the right angled triangle *KLM* shown in the figure, *KM* is the hypotenuse while *KL* and *LM* are the remaining sides which include the right angle.

Area of the square drawn on the side $KL = KL^2$

Area of the square drawn on the side $LM = LM^2$

Area of the square drawn on the hypotenuse $KM = KM^2$



Therefore, according to Pythagoras' Theorem,

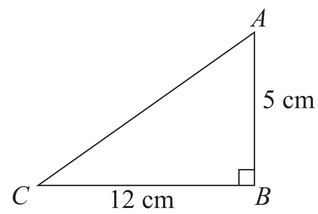
$$\boxed{KL^2 + LM^2 = KM^2}$$

Let us now consider how calculations are performed using Pythagoras' Theorem.

Example 1

In the right angled triangle *ABC*, $\hat{B} = 90^\circ$, $AB = 5$ cm and $BC = 12$ cm. Find the length of the side *AC*.

According to Pythagoras' Theorem,



$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= 5^2 + 12^2 \\ &= 25 + 144 \\ &= 169 \end{aligned}$$

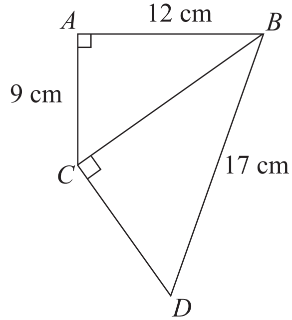
$$\therefore \quad AC = \sqrt{169}$$
$$= 13$$

$\therefore$ the length of side *AC* is 13cm.

Example 2

Find the length of *CD* based on the information given in the figure.

According to the figure, Pythagoras' Theorem can be applied to the right angled triangle *ABC*.



$$\therefore \; BC^2 = AB^2 + AC^2$$
$$= 12^2 + 9^2$$
$$= 144 + 81$$
$$= 225$$
$$\therefore \; BC = \sqrt{225}$$
$$= 15$$

Applying Pythagoras' Theorem to the right angled triangle *BCD*,

$$CD^2 + BC^2 = BD^2$$
$$CD^2 + 15^2 = 17^2$$
$$CD^2 + 225 = 289$$
$$\therefore \; CD^2 = 289 - 225$$
$$= 64$$
$$\therefore \; CD = 8$$

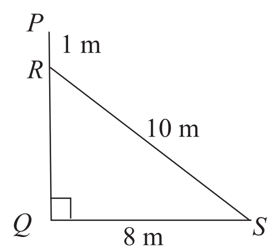
$\therefore$ the length of *CD* is 8 cm.

Now let us consider how Pythagoras' Theorem can be used to solve practical problems.

Example 3

One end of a wire is tied to a ring fastened at a point 1m below the top of a vertical utility pole, while the other end is tied to a ring fastened on the ground, 8 m away from the foot of the pole. The length of the wire between the two rings is 10 m. Find the height of the utility pole. (Assume that the wire is stretched.)

Let us draw the figure according to the given information.



As the pole *PQ* is vertical, it makes a right angle with the horizontal ground. Therefore $P\hat{Q}S = 90^\circ$.
Since *QRS* is a right angled triangle, according to Pythagoras' Theorem,

$$
\begin{aligned}
QR^2 + QS^2 &= RS^2\\
QR^2 + 8^2 &= 10^2\\
QR^2 + 64 &= 100\\
\therefore\ QR^2 &= 100 - 64\\
QR^2 &= 36\\
\therefore\ QR &= 6
\end{aligned}
$$

$\therefore$ Height of the pole $= QR + PR$

$= 6 + 1$

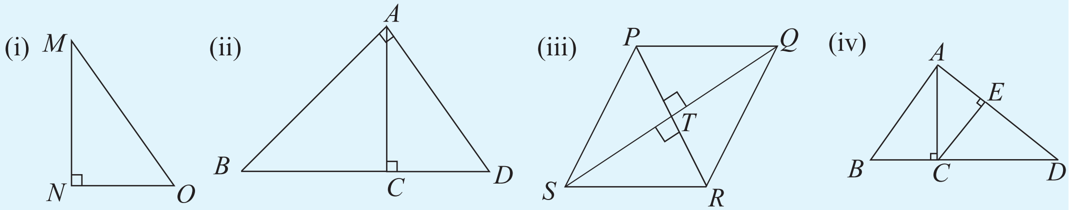
$= 7$

$\therefore$ Height of the pole is 7 m.

Now do the following exercise using Pythagoras' Theorem.

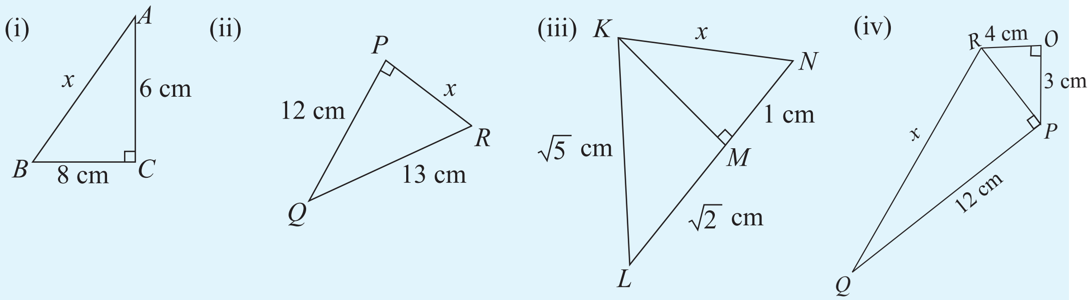
Exercise 17.1

1. Fill in the blanks using the information in the figure.



(i)	(ii)	(iii)	(iv)
$MO^2 = ... + ...$	$BD^2 = ...\ + ...$	$PQ^2 = ... + ...$	$AB^2 = ... + AC^2$
	$....\ = AC^2 + CD^2$	$QR^2 = ... + ...$	$.... = AE^2 + EC^2$
	$AB^2 = AC^2 + ...$		$AD^2 = AC^2 + ...$

2. Find the value of x in each of the right angled triangles given below.



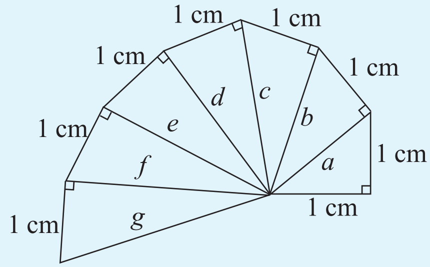
3. In the equilateral triangle ABC, D is the foot of the perpendicular drawn from the vertex A to the side BC. If the length of a side of the triangle is 2 cm, find the length of AD (Express the answer as a surd.)

4. The location Q is reached from the location P on the horizontal ground, by travelling 15m to the North from P and then 8m to the East.

 (i) Draw a sketch based on the above information.
 (ii) Find the distance PQ.

5. The lengths of the diagonals of a rhombus are 12 m and 16 m. Find the length of each side of the rhombus.

6. The figure illustrates the special creation Archimedes' spiral. By considering the right angled triangles in the figure, find the lengths a, b, c, d, e, f and g, using the given measurements. (Express the answers in surd form)

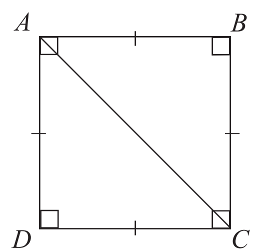


17.2 Further applications of Pythagoras' Theorem

Now let us consider how riders related to Pythagoras' Theorem are proved.

Example 1

$ABCD$ is a square. Prove that $AC^2 = 2AB^2$.



Proof : ABC is a right angled triangle since $A\hat{B}C = 90^\circ$

Applying Pythagoras' Theorem to the triangle ABC,

$AC^2 = AB^2 + BC^2$

$AC^2 = AB^2 + AB^2$ ($AB = BC$, sides of a square)

$\therefore \quad \underline{\underline{AC^2 = 2AB^2}}$

Example 2

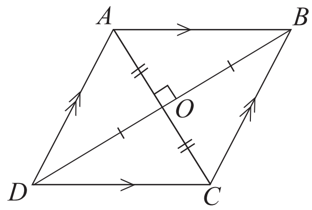
In the rhombus $ABCD$, the diagonals AC and BD intersect at O. Prove that $AC^2 + BD^2 = 4\,AB^2$.

Proof: Since $ABCD$ is a rhombus, the diagonals bisect each other perpendicularly. (See figure)

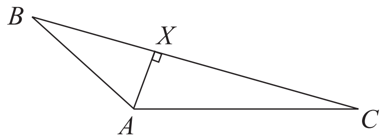
$\therefore \quad A\hat{O}B = 90^\circ$, $AO = OC$ and $BO = OD$.

According to Pythagoras' Theorem; in the right angled triangle AOB,

$$AO^2 + OB^2 = AB^2$$
$$(\tfrac{1}{2}AC)^2 + (\tfrac{1}{2}BD)^2 = AB^2$$
$$\tfrac{1}{4}AC^2 + \tfrac{1}{4}BD^2 = AB^2$$
$$\tfrac{1}{4}(AC^2 + BD^2) = AB^2$$
$$\therefore \quad \underline{\underline{AC^2 + BD^2 = 4\,AB^2}}$$



Example 3



In the triangle ABC, $B\hat{A}C$ is an obtuse angle.
AX is drawn from A,
perpendicular to BC. Prove that $AB^2 - AC^2 = BX^2 - CX^2$

Proof:

In the right angled triangle AXB, according to Pythagoras' Theorem

$AB^2 = AX^2 + BX^2$ —— ①

In the right angled triangle AXC, according to Pythagoras' Theorem,

$AC^2 = AX^2 + CX^2$ —— ②

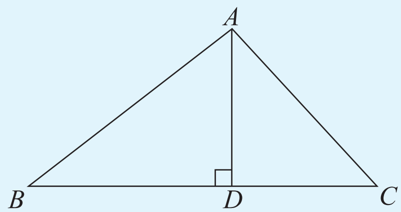
① − ② ; $AB^2 - AC^2 = AX^2 + BX^2 - (AX^2 + CX^2)$

$= AX^2 + BX^2 - AX^2 - CX^2$

$= \underline{\underline{BX^2 - CX^2}}$

Prove the riders in the exercise given below as illustrated in the above examples.

Exercise 17.2

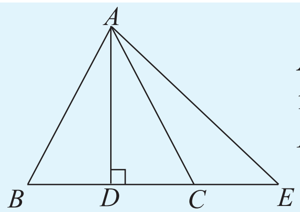


1. AD is perpendicular to BC in the triangle ABC. (See figure)
If $AD = DC$, prove that $AB^2 = BD^2 + DC^2$.

2. AD is perpendicular to BC in the triangle ABC. Prove that $AB^2 + CD^2 = AC^2 + BD^2$.

3. AD is perpendicular to BC in the equilateral triangle ABC. Prove that $4\,AD^2 = 3\,BC^2$.

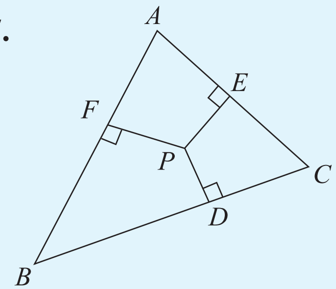
for free distribution.

4. 

AD is perpendicular to BC in the equilateral triangle ABC in the figure. BC has been produced to E such that $DC = CE$. Prove that $AE^2 = 7\,EC^2$.

5. The diagonals of the quadrilateral $ABCD$ bisect each other perpendicularly at O. Prove that $AB^2 + CD^2 = AD^2 + BC^2$.

6. O is a point within the rectangle $ABCD$. Prove that $AO^2 + CO^2 = BO^2 + DO^2$ (Hint: Draw a parallel line through O to any side of $ABCD$.)

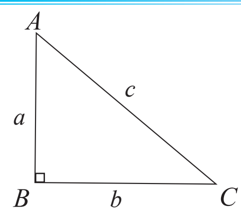
7. 

P is a point within the triangle ABC. The perpendiculars drawn from the point P to the sides BC, AC and AB meet these sides at D, E and F respectively.
Prove that,

(i) $BP^2 - PC^2 = BD^2 - DC^2$ and
(ii) $BD^2 + CE^2 + AF^2 = CD^2 + AE^2 + BF^2$

8. The two squares $ABXY$ and $BCPQ$ lie on the same side of the straight line ABC. Prove that $PX^2 + CY^2 = 3\,(AB^2 + BC^2)$

17.3 Pythagorean triples



In the right angled triangle ABC in the figure, if the lengths of the sides which include the right angle are a and b units, and the length of the hypotenuse is c units, then we know that $a^2 + b^2 = c^2$ according to Pythagoras' Theorem. Values of a, b and c which satisfy the equation

$a^2 + b^2 = c^2$ are known as Pythagorean triples.

Since $3^2 + 4^2 = 5^2$, we obtain that (3, 4, 5) is a Pythagorean triple. Any multiple of the triple (3, 4, 5) is also a Pythagorean triple.

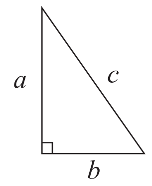
Eg: Multiplying each value of the triple (3, 4, 5) by 2 we obtain (6, 8, 10). Since $6^2 + 8^2 = 10^2$, (6, 8, 10) is a Pythagorean triple.

Multiplying each value of the triple (3, 4, 5) by 3 we obtain (9, 12, 15). Since $9^2 + 12^2 = 15^2$, (9, 12, 15) is also a Pythagorean triple.
There are Pythagorean triples apart from the multiples of (3, 4, 5).

Eg: Since $5^2 + 12^2 = 13^2$, (5, 12, 13) is a Pythagorean triple.
Since $8^2 + 15^2 = 17^2$, (8, 15, 17) is a Pythagorean triple.

A mathematician named Euclid introduced "parametric equations" to find Pythagorean triples. Given any two numbers x and y, if $a = x^2 - y^2$, $b = 2xy$ and $c = x^2 + y^2$, then $a^2 + b^2 = c^2$, and hence (a, b, c) is a Pythagorean triple.

Eg: $x = 6$, and $y = 5$, then $a = x^2 - y^2 = 6^2 - 5^2 = 11$
$b = 2xy = 2 \times 6 \times 5 = 60$
$c = x^2 + y^2 = 6^2 + 5^2 = 61$.
Therefore (11, 60, 61) is a Pythagorean triple.

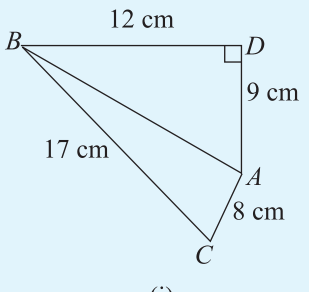


Exercise 17.3

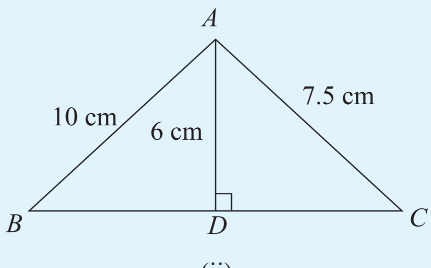
1. The following triples are the lengths of the sides of two triangles. Select the triangle which is a right angled triangle and write down the corresponding Pythagorean triple.

(i) (8, 15, 17) (ii) (14, 18, 25)

2. Based on the measurements given in figures (i) and (ii), show that $B\hat{A}C$ is a right angle in each figure.



(i)



(ii)

for free distribution.

3. By completing the table given below, find the Pythagorean triples corresponding to the given pairs of values. Verify your answers.

x	y	x^2	y^2	a	b	c	Pythagorean triple
				$x^2 - y^2$	$2xy$	$x^2 + y^2$	
2	1						
5	4						
4	3						
6	5						
7	5						

Miscellaneous Exercise

1. The chord AB of the circle with centre O, which lies at a distance of 9 cm from O, is of length 24 cm. Find the radius of the circle.

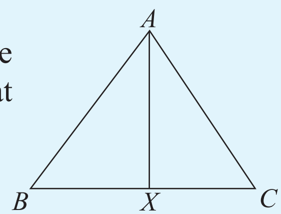
2. Construct the triangle ABC where $AB = 2$cm, $BC = 3$cm and $\hat{B}$ is a right angle. Using the triangle you constructed, find the value of $\sqrt{13}$ to the first decimal place.

3. Construct straight line segments of the lengths given below.

(i) $\sqrt{8}$ cm (ii) $\sqrt{10}$ cm (iii) $\sqrt{41}$ cm

4. ABC is an equilateral triangle. D is the midpoint of AB and E is the midpoint of CD. Prove that $16\,AE^2 = 7\,AB^2$.

5. In the triangle ABC, $\hat{B}$ is an acute angle. The foot of the perpendicular dropped from A to BC is X. Prove that $AC^2 = AB^2 + BC^2 - 2\,BC.BX$



18 Trigonometry

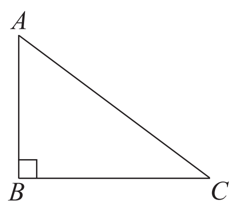
By studying this lesson you will be able to,

- identify the trigonometric ratios sine, cosine and tangent,
- perform calculations related to triangles using the sines, cosines and tangents tables,
- use the scientific calculator to examine the accuracy of the solutions to trigonometry problems.

18.1 Right Angled Triangles

We know that we can use Pythagoras' relationship to find the length of a side of a right angled triangle (right triangle) when the lengths of the other two sides are given.

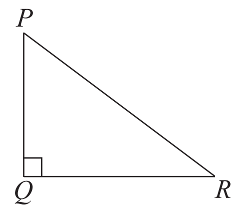
Pythagoras' relationship cannot be used to find the lengths of the remaining sides of a triangle when the length of one side of a right triangle and the magnitude of an angle other than the right angle are given. To identify a method to do this, let us first see how the sides of a right angled triangle are named.



The angle $\hat{B}$ of the right angled triangle ABC is a right angle. Therefore $\hat{A}$ and $\hat{C}$ are acute angles. The side AC which is opposite the right angle $\hat{B}$ is defined as the hypotenuse of the triangle. When the angle $\hat{C}$ is considered, then AB which is directly opposite $\hat{C}$ is called the "opposite side". Moreover, the side BC which is one of the arms of the angle $\hat{C}$, the other being the hypotenuse, is called the "adjacent side".

Accordingly, when $\hat{A}$ is considered, as above, we have that BC is the "opposite side" and AB is the "adjacent side".

Thus, for the triangle PQR in the figure,



the hypotenuse $= PR$

when $Q\hat{R}P$ is considered, the opposite side $= PQ$

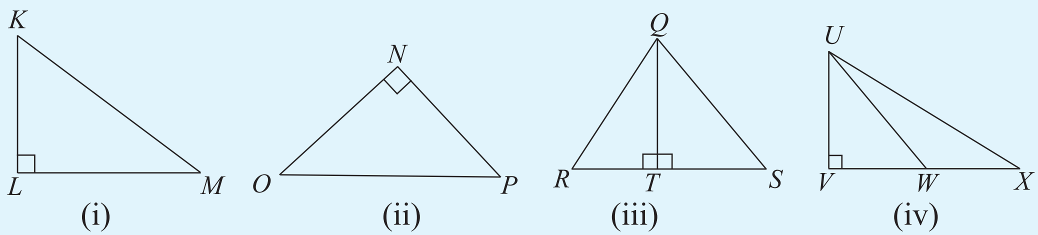
the adjacent side $= QR$

when $Q\hat{P}R$ is considered, the opposite side $= QR$

the adjacent side $= PQ$.

Exercise 18.1

1. Complete the following table using the given figures.



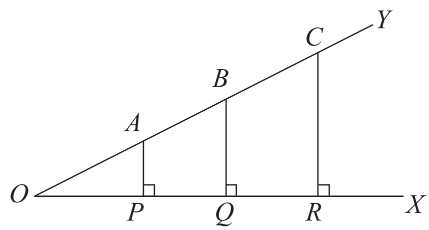
	Right angled triangle	Hypotenuse	Angle considered	Opposite side	Adjacent side
(i)	KLM	KM	$L\hat{K}M$ $L\hat{M}K$		
(ii)	PNO		$N\hat{O}P$ $O\hat{P}N$		
(iii)	QRT QTS		$R\hat{Q}T$ $T\hat{Q}S$		
(iv)	UVX UVW		$V\hat{U}X$ $U\hat{W}V$		

18.2 Trigonometric Ratios

Do the following exercise to investigate the relationship between two sides of a right angled triangle and an angle of the triangle.

Activity

- Draw the angle $X\hat{O}Y$ (= 30°) such that the arms XO and OY of this angle are of length 11 cm each.
- Mark the points A, B and C on the side OY at distances of 2 cm, 4 cm and 7 cm respectively from O.
- Using a set square or by some other method, draw lines perpendicular to OX through the points A, B and C and name the points that these perpendiculars meet OX as P, Q and R respectively.
- Then you will obtain a figure similar to the one given below.



- Measure the lengths of the sides of each of the right angled triangles that are obtained and complete the following table. (Write down all measured values and calculated values to the nearest first decimal place).

Right angled triangle	Hypotenuse	Side opposite 30°	Side adjacent to 30°	$\frac{\text{Opposite side}}{\text{Hypotenuse}}$	$\frac{\text{Adjacent side}}{\text{Hypotenuse}}$	$\frac{\text{Opposite side}}{\text{Adjacent side}}$
AOP	2	1	1.7	$\frac{1}{2} = 0.5$	$\frac{1.7}{2} = 0.9$	$\frac{1}{1.7} = 0.6$
BOQ						
COR						

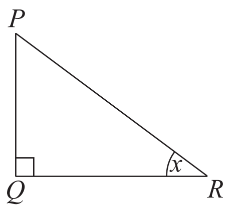
According to the above table which was prepared based on the measurements that were obtained from the activity,
for the angle 30°, from all three triangles we obtain

$$\frac{\text{opposite side}}{\text{hypotenuse}} = 0.5$$

$$\frac{\text{opposite side}}{\text{adjacent side}} = 0.6$$

$$\frac{\text{adjacent side}}{\text{hypotenuse}} = 0.9$$

Notice that the reason why a constant value is obtained for these ratios of sides of the right angled triangles in the above figure, is because the triangles are all equiangular. The above ratios which are defined for right angled triangles are called trigonometric ratios. These trigonometric ratios are called the sine of the angle 30°, the tangent of the angle 30° and the cosine of the angle 30°, depending on the sides connected with them. "sin" is used for sine, "tan" is used for tangent and "cos" is used for cosine. Accordingly, the sine of the angle 30° is written as sin 30°, the tangent of the angle 30° is written as tan 30° and the cosine of the angle 30° is written as cos 30°.



Now let us write the trigonometric ratios for the triangle *PQR* in the figure using the notation given above.

In terms of x;

$$\sin x = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{PQ}{PR}$$

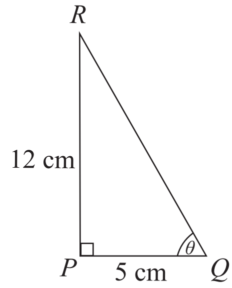
$$\cos x = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{QR}{PR}$$

$$\tan x = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{PQ}{QR}$$

Now let us see how calculations are done using these three trigonometric ratios by considering some examples.

Example 1

In the triangle *PQR* in the figure, $\hat{P}$ is a right angle. $PQ = 5$ cm and $PR = 12$ cm. $P\hat{Q}R = \theta$.



(i) Find the length of the side QR.

(ii) Find the following values.

(a) $\sin\theta$ **(b)** $\cos\theta$ **(c)** $\tan\theta$

(i) According to Pythagoras' relationship,

$$QR^2 = PQ^2 + PR^2$$
$$= 5^2 + 12^2$$
$$= 25 + 144$$
$$\therefore \quad QR = \sqrt{169}$$
$$= 13$$

$\therefore$ the length of the side QR is 13 cm.

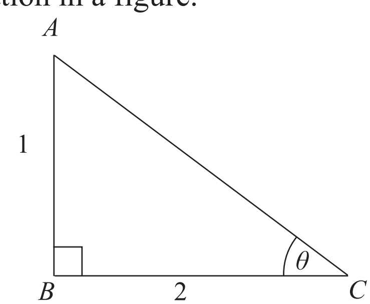
(ii) **(a)** $\sin\theta = \dfrac{PR}{QR} = \dfrac{12}{13} = \underline{\underline{0.9230}}$ **(b)** $\cos\theta = \dfrac{PQ}{QR} = \dfrac{5}{13} = \underline{\underline{0.3846}}$ **(c)** $\tan\theta = \dfrac{PR}{PQ} = \dfrac{12}{5} = \underline{\underline{2.4}}$

Example 2

If $\tan\theta = \dfrac{1}{2}$, find the values of $\sin\theta$ and $\cos\theta$.

If $\tan\theta = \dfrac{1}{2}$, the side opposite θ is of length 1 unit and the side adjacent to θ is of length 2 units.

Let us represent this information in a figure.



Then, according to Pythagoras' relationship,

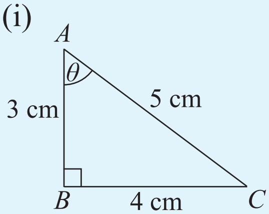
$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= 1^2 + 2^2 \\ &= 5 \\ \therefore \quad AC &= \sqrt{5} \end{aligned}$$

Then, $\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \underline{\underline{\dfrac{1}{\sqrt{5}}}}$ $\qquad$ $\cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \underline{\underline{\dfrac{2}{\sqrt{5}}}}$

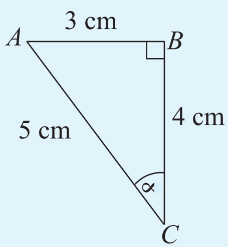
Exercise 18.2

1. Fill in the blanks under each figure, based on the information given in each of the following figures.

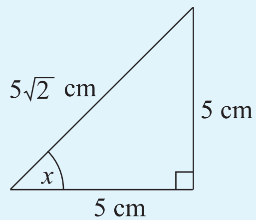
(i)



(ii)



(iii)



$\sin\theta =$ ....
$\cos\theta =$ ....
$\tan\theta =$ ....

$\sin\alpha =$ ....
$\cos\alpha =$ ....
$\tan\alpha =$ ....

$\sin x =$ ....
$\cos x =$ ....
$\tan x =$ ....

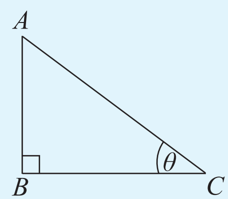
2. If $\sin\theta = \dfrac{5}{13}$, find the value of (i) $\tan\theta$ (ii) $\cos\theta$.

3. In the triangle ABC in the figure, $\hat{B}$ is a right angle and $\hat{C} = \theta$.

(i) Express $B\hat{A}C$ in terms of θ.

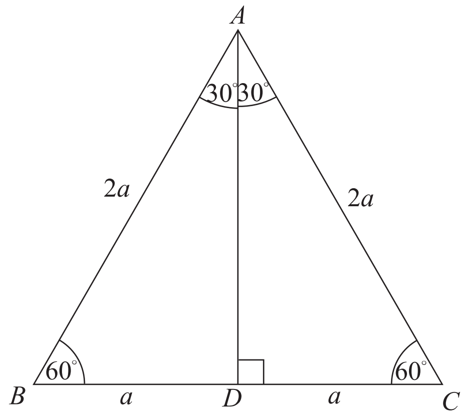
(ii) Show that $\sin\theta = \cos(90^\circ - \theta)$.

(iii) Show that $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$.



18.3 The trigonometric ratios of the angles of magnitude 30°, 45° and 60°

By considering an equilateral triangle of side length $2a$, the trigonometric ratios for the angles 30° and 60° can be found.



The figure depicts an equilateral triangle ABC. Its vertex angles are each of magnitude 60°. We know that when the perpendicular AD from the vertex A to the side BC is drawn, D is the midpoint of BC and $B\hat{A}C$ is bisected by AD. Then $B\hat{A}D = 30^\circ$.

Let us find the length of the side AD of the right angled triangle ABD in terms of a.

According to Pythagoras' theorem;

$$
\begin{aligned}
BD^2 + AD^2 &= AB^2 \\
a^2 + AD^2 &= (2a)^2 \\
AD^2 &= 4a^2 - a^2 \\
&= 3a^2 \\
AD &= \sqrt{3}\,a
\end{aligned}
$$

If we consider the right triangle ABD,

$$\sin 60^\circ = \frac{AD}{AB} = \frac{\sqrt{3}\,a}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2}$$

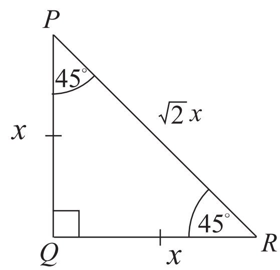
$$\tan 60^\circ = \frac{AD}{BD} = \frac{\sqrt{3}\,a}{a} = \sqrt{3}$$

If we consider the right angled triangle *ABD,*

$$\sin 30^\circ = \frac{BD}{AB} = \frac{a}{2a} = \frac{1}{2} \qquad \cos 30^\circ = \frac{AD}{AB} = \frac{\sqrt{3}a}{2a} = \frac{\sqrt{3}}{2} \qquad \tan 30^\circ = \frac{BD}{AD} = \frac{a}{\sqrt{3}a} = \frac{1}{\sqrt{3}}$$

Let us now use the isosceles right angled triangle *PQR* in the given figure to obtain the trigonometric ratios for the angle 45°. Let us take the length of the sides of the triangle that include the right angle to be x.

Then, according to Pythagoras' theorem,



$$PR^2 = x^2 + x^2$$
$$= 2x^2$$
$$\therefore \quad PR = \sqrt{2}x$$

Accordingly,

$$\sin 45^\circ = \frac{PQ}{PR} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}} \qquad \cos 45^\circ = \frac{QR}{PR} = \frac{x}{\sqrt{2}x} = \frac{1}{\sqrt{2}} \qquad \tan 45^\circ = \frac{PQ}{QR} = \frac{x}{x} = 1$$

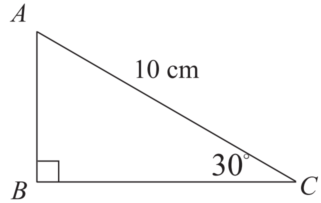
The trigonometric ratios obtained for the angles of magnitude 30°, 60° and 45° are given in the following table.

	30°	45°	60°
sin	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
tan	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Example 1

In the right angled triangle ABC, $\hat{B} = 90^\circ$, $A\hat{C}B = 30^\circ$ and $AC = 10$ cm. Find the lengths of AB and BC.

According to the figure,



$$\sin 30^\circ = \frac{AB}{AC}$$

$$\frac{1}{2} = \frac{AB}{10}$$

$$AB = 5$$

$\therefore$ the length of the side AB is 5 cm.

$$\cos 30^\circ = \frac{BC}{AC}$$

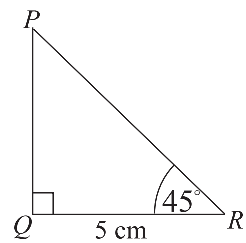
$$\frac{\sqrt{3}}{2} = \frac{BC}{10}$$

$$\therefore \quad BC = 5\sqrt{3}$$

$\therefore$ the length of the side BC is $5\sqrt{3}$ cm.

Example 2

Find the length of the hypotenuse of the right angled triangle PQR.



$$\cos 45^\circ = \frac{QR}{PR}$$

$$\frac{1}{\sqrt{2}} = \frac{5}{PR}$$

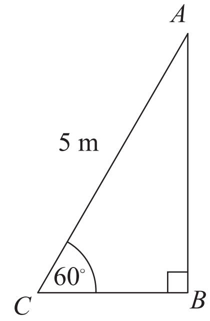
$$\therefore \quad PR = 5\sqrt{2}$$

$\therefore$ the length of the hypotenuse is $5\sqrt{2}$ cm.

Example 3

A 5 m long ladder is kept leaning against a vertical wall such that the angle between the ladder and the horizontal is 60°. At what height above the horizontal ground does the top of the ladder touch the wall?

Since the angle between the vertical wall and the horizontal ground is 90°, $A\hat{B}C = 90^\circ$ in the figure.



In the right angled triangle *ABC*,

$$\sin 60^\circ = \frac{AB}{AC}$$

$$\therefore \ \frac{\sqrt{3}}{2} = \frac{AB}{5}$$

$$\therefore \ AB = \frac{5\sqrt{3}}{2}$$

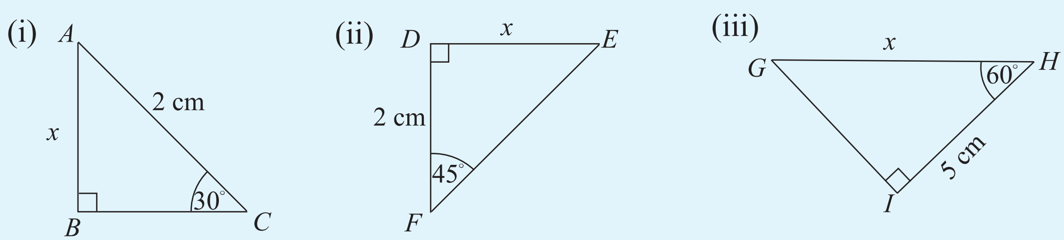
$= 4.325$ (By taking $\sqrt{3} = 1.73$)

$\therefore$ the top of the ladder touches the wall at a height of 4.33 m above the ground.

Now do the following exercise by using the values in the table in section 18.3.

Exercise 18.3

1. Find the length x in each of the triangles given below using the information given in the triangle.



2. Find the value of each of the following expressions using the information in the table in section 18.3.

a. $\sin 30^\circ + \cos 60^\circ$

b. $\sin 45^\circ + \cos 45^\circ + \tan 60^\circ$

c. $\sin 60^\circ + \cos 30^\circ + \tan 60^\circ$

d. $\cos 60^\circ + \sin 30^\circ + \tan 60^\circ$

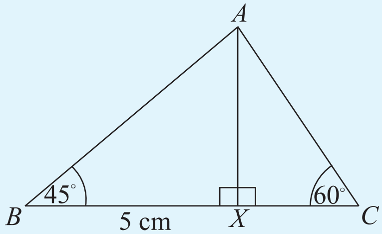
3. Verify the following.

(i) $\sin 30^\circ \ \cos 60^\circ + \cos 30^\circ \sin 60^\circ = 1$

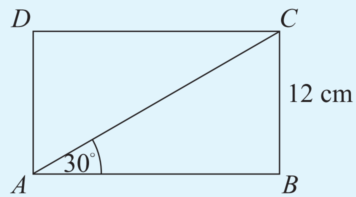
(ii) $\cos 60^\circ \ \cos 30^\circ - \sin 60^\circ \ \sin 30^\circ = 0$

(iii) $\tan 30^\circ = \dfrac{\tan 60^\circ - \tan 30^\circ}{1 + \tan 60^\circ \tan 30^\circ}$

4. Based on the information in the figure,
(i) find the length of AX.
(ii) find the length of AC.
(Take $\sqrt{3} = 1.7$)



5. Find the length of the diagonal of the rectangle $ABCD$ in the figure if the length of the side BC is 12 cm.



6. To keep an antenna post vertical, one end of a stretched wire has been attached to a point which is 50 cm below the top of the post, while the other end has been attached to a wedge which is firmly fixed to the horizontal ground, 5 m away from the foot of the post. The angle between the horizontal ground and the wire is 30° .

(i) Represent this information in a sketch.
(ii) Find the height of the post by taking $\sqrt{3} = 1.7$

18.4 The Trigonometric Tables

So far we have considered only the trigonometric ratios of the angles 30°, 45° and 60°. However, there are trigonometric ratios for all angles from 0° to 90°. The trigonometric ratios corresponding to these angles have been tabulated. Separate tables have been prepared for sine, cosine and tangent. The "degree" which is a unit of angles can be divided into a smaller unit called "minute". One degree is equal to 60 minutes. i.e., $1^\circ = 60'$. Trigonometric ratio values for angles in degrees and minutes are provided in the trigonometric tables.

In all three tables, the sines, the cosines and the tangents, the first column has the angles from 0 to 90°. The following is a part of the Tangents Table.

ප්‍රකෘති ටැංජන
இயற்கைத் தான்சன்கள்
NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences (මධ්‍යක අන්තරය / இடை வித்தியாசங்கள்)								
0°	0·0000	0·0029	0·0058	0·0087	0·0116	0·0145	0·0175	89°	3	6	9	12	15	17	20	23	26
1	·0175	·0204	·0233	·0262	·0291	·0320	·0349	88	3	6	9	12	15	17	20	23	26
2	·0349	·0378	·0407	·0437	·0466	·0495	·0524	87	3	6	9	12	15	18	20	23	26
3	·0524	·0553	·0582	·0612	·0641	·0670	·0699	86	3	6	9	12	15	18	20	23	26
4	·0699	·0729	·0758	·0787	·0816	·0846	·0875	85	3	6	9	12	15	18	21	23	26

In the trigonometric ratios tables, the first column has the angles from 0° to 90° (Since only a portion of the table is given here, only the angles from 0° to 4° are shown). The parts of a degree which are minutes are given in the first row of the table as $0'$, $10'$, $20'$ etc., and as $1'$, $2'$, ...$9'$ in the Mean Differences column. When finding the trigonometric ratio of an angle, the value in the relevant row and column, and sometimes the value in the Mean Differences column is used as is done when using the logarithms table.

Now let us consider each of the above mentioned trigonometric tables separately.

The Tangents Table

The ratios in the Tangents Table start with 0.0000, increase gradually, exceed 1.0000 and become extremely large as the angle approaches the magnitude 90°. Below is another portion of the Tangents Table.

ප්‍රකෘති ටැංජන
இயற்கைத் தான்கள்கள்
NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									Mean Differences (මධ්‍යක අන්තරය / இடை வித்தியாசங்கள்)								
42	.9004	.9057	.9110	.9163	.9217	.9271	.9325	47	5	11	16	21	27	32	37	43	48
43	.9325	.9380	.9435	.9490	.9545	.9601	.9657	46	6	11	17	22	28	33	39	44	50
44	.9657	.9731	.9770	.9827	.9884	.9942	1.0000	45	6	11	17	23	29	34	40	46	51
45°	1.0000	1.0058	1.0117	1.0176	1.0235	1.0295	1.0355	44°	6	12	18	24	30	36	41	47	53
46	$\boxed{.0355}$	.0416	.0477	.0538	.0599	.0661	.0724	43	6	12	18	25	31	37	43	49	55
47	.0724	.0786	.0850	.0913	.0977	.1041	.1106	42	6	13	19	26	32	38	45	51	57
48	.1106	.1171	$\boxed{.1237}$	.1303	.1369	.1436	.1504	41	7	13	20	27	33	40	46	53	60
49	.1504	.1571	.1640	.1708	.1778	$\boxed{.1847}$	.1918	40°	7	14	21	28	34	41	$\boxed{48}$	55	62

Let us first find the value of tan 43°. The value corresponding to tan 43° appears in the row which contains 43° and the column which contains $0'$. Accordingly, tan $43^\circ = 0.9325$.

Now let us find the value of tan $48^\circ\ 20'$

We need to move along the row containing 48° until we arrive at the column containing 20′. Take the value.1237 which is in this position. Since the whole number part of the number 1.0117 at the top of the column containing 20′ is 1, all the numbers along that column should also have a whole number part equal to 1. (The reason for writing the relevant whole number part only in the first row is to preserve the clarity of the tables). Accordingly, the value of tan 48° 20′ is 1.1237.

Let us find the value of tan 49° 57′ similarly. Here the value of tan 49° 50′ needs to be found first.

$$\tan 49^\circ\ 50' = 1.1847$$

To find the tangent value corresponding to 49° 57′, we need to add the value from the Mean Differences column, corresponding to 7′, which is 0.0048, to the value 1.1847 (as a convention, the mean difference is considered to be a value with four decimal places with only the non-zero part given in the tables).

Then we obtain

$$\begin{aligned}\tan 49^\circ\ 57' &= 1.1847 + 0.0048\\ &= 1.1895\end{aligned}$$

Example 1

(i) $\tan 34^\circ\ 30' = 0.6873$

(ii) $\tan 44^\circ\ 42' = 0.9884 + 0.0011$
$\qquad = 0.9895$

(iii) $\tan 79^\circ\ 25' = 5.309 + 0.044$
$\qquad = 5.353$

When it is required to find an angle using the tables when a trigonometric ratio of the angle is known, a procedure similar to that followed in finding the antilog of a value using the logarithms table is used.

Let us find θ such that $\tan \theta = 1.1054$

ප්‍රකෘති ටැංජන
இயற்கைத் தான்சன்கள்
NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
45°	1·0000	1·0058	1·0117	1·0176	1·0235	1·0295	1·0355	44	6	12	18	24	30	36	41	47	53
46	·0355	·0416	·0477	·0538	·0599	·0661	·0724	43	6	12	18	25	31	37	43	49	55
47	·0724	·0786	·0850	·0913	·0977	·1041	·1106	42	6	13	19	26	32	38	45	51	57
48	·1106	·1171	·1237	·1303	·1369	·1436	·1504	41	7	13	20	27	33	40	46	53	60
49	·1504	·1571	·1640	·1708	·1778	·1847	·1918	40°	7	14	21	28	34	41	48	55	62

Find the value in the table which is closest to 1.1054 but less than it. This value is 1.1041. From the table it can be seen that the angle corresponding to this value is $47^\circ\ 50'$. We need to add 0.0013 to 1.1041 to obtain the value 1.1054. Therefore, the number of minutes corresponding to 0.0013 (that is, the value 13 in the Mean Differences column) has to be added to $47^\circ\ 50'$ to obtain the correct angle. This is 2 minutes as highlighted in the table. Therefore, the angle of which the tangent value is 1.1054 is $47^\circ\ 50' + 2' = 47^\circ\ 52'$. Therefore, $\theta = 47^\circ\ 52'$.

Example 2

(i) If $\tan\theta = 0.3706$

$\theta = 20^\circ\ 20'$

(ii) If $\tan\theta = 0.4774$

$\theta = 25^\circ\ 30' + 1'$

$= 25^\circ\ 31'$

(iii) If $\tan\theta = 0.8446$

$\theta = 40^\circ\ 11'$

The Sines Table

This table contains values from 0.0000 to 1.0000. As in the Tangents Table, the first column contains the angles from 0° to 90°. In the first row right at the top of the table, the minute values, namely $0'$, $10'$, $20'$ etc., and in the Mean Differences column, the values $1'$, $2'$, ... $9'$ appear. This table is used in the same way that the Tangents Table is used.

Note: Although the values in the Tangents Table start from 0 and increase to very great values, the Sines Table contains only values from 0 to 1. The reason for this is because the sine value of an angle in a triangle always takes a value from 0 to 1.

Let us find the value of $\sin 33^\circ\ 27'$ using the table.

ප්‍රකෘති සයින
இயற்கைச் சைன்கள்
NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									මධ්‍යක අන්තරය இடை வித்தியாசங்கள் Mean Differences								
30°	0·5000	0·5025	0·5050	0·5075	0·5100	0·5125	0·5150	59	2	5	8	10	13	15	18	20	23
31	·5150	·5175	·5200	·5225	·5250	·5275	·5299	58	2	5	7	10	12	15	17	20	22
32	·5299	·5324	·5348	·5373	·5398	·5422	·5446	57	2	5	7	10	12	15	17	20	22
33	·5446	·5471	·5495	·5519	·5544	·5568	·5592	56	2	5	7	10	12	15	17	19	22
34	·5592	·5616	·5640	·5664	·5688	·5712	·5736	55	2	5	7	10	12	14	17	19	22

First note that $\sin 33^\circ 20' = 0.5495$. To obtain the value corresponding to the remaining $7'$, move along the row containing 33° and find the value corresponding to $7'$ from the Mean Differences column. This value is 0.0017. Add this value to 0.5495 to obtain the value of $\sin 33^\circ 27'$.

That is, $\sin 33^\circ 27' = 0.5495 + 0.0017 = 0.5512$.

Example 3

(i) $\sin 75^\circ\, 44' = 0.9689 + 0.0003$
$= 0.9692$

(ii) $\sin 45^\circ\, 34' = 0.7133 + 0.0008$
$= 0.7141$

(iii) $\sin 39^\circ\, 50' = 0.6406$

Now let us use the table to find the angle corresponding to a given sine value. This is also done in the same way that we found the angle corresponding to a given tangent value.

Let us find the angle θ if $\sin \theta = 0.5075$. It appear in the raw 30° and $30'$ column. accordingly if $\sin \theta = 0.5075$, the $\theta = 30^\circ\, 30'$.

Now let us find another angle using the table.
The angle θ if $\sin \theta = 0.5277$. Since 0.5277 does not appear in the table, consider the value 0.5275 which is the closest value in the table that is less than the given value. The angle corresponding to this is $31^\circ\, 50'$. Consider the values in the Mean Differences column along the same row to find the number of minutes corresponding to the remaining 0.0002. The number of minutes corresponding to the value 2 in the Mean Differences column is $1'$. $\therefore$ The angle of which the sine value is 0.5277 is $31^\circ\, 51'$.

That is, if $\sin \theta = 0.5277$, then $\theta = 31^\circ\, 51'$.

Example 4

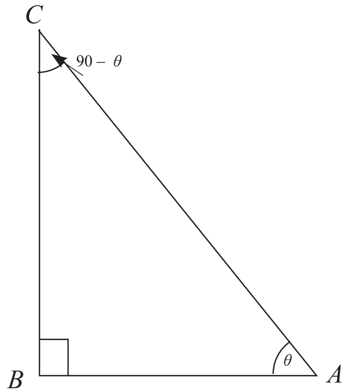
(i) If $\sin \theta = 0.5831$
then $\theta = 35^\circ\, 40'$

(ii) If $\sin \theta = 0.7036$
then $\theta = 44^\circ\, 43'$

(iii) If $\sin \theta = 0.9691$
then $\theta = 75^\circ\, 43'$

Cosines

Consider the following triangle.



The above triangle is a right angled triangle with $A\hat{B}C = 90^\circ$ Let us take $B\hat{A}C = \theta$. Then, since the sum of the angles of a triangle is 180°, $A\hat{C}B = 90^\circ - \theta$.

The sum of the angles $A\hat{C}B$ and $B\hat{A}C$ is 90°. You have learnt previously that such pairs of angles are called complementary angles.

If we consider the triangle ABC,

$$\cos\theta = \frac{\text{side adjacent to } \hat{A}}{\text{hypotenuse}} = \frac{AB}{AC}.$$

This can also be written as,

$$\sin(90^\circ - \theta) = \frac{\text{side opposite to } \hat{C}}{\text{hypotenuse}} = \frac{AB}{AC}.$$

Accordingly, $\cos\theta = \sin(90 - \theta)$.

This relationship can be used to find the cosine value of an angle in a triangle.

Example 1

Find the value of $\cos 58^\circ$.

$\cos 58^\circ = \sin(90^\circ - 58^\circ)$ (according to the relationship that was obtained)
$= \sin 32^\circ$
$= \underline{\underline{0.5299}}$ (according to the part of the table given above)

Example 2

Find the value of cos $56^\circ\ 18'$.

First let us find the value of $90 - 56^\circ\ 18'$. It is $33^\circ\ 42'$.
Therefore $\cos 56^\circ\ 18' = \sin (90 - 56^\circ\ 18') = \sin 33^\circ\ 42'$

$$= \underline{\underline{0.5549}}$$

We can similarly find the angle when the cosine value has been given. Let us consider an example.

Example 3

Find the value of θ if $\cos \theta = 0.5175$.
Let us write this as $\sin (90 - \theta) = 0.5175$.
Next let us find the angle of which the sine value is 0.5175. According to the table it is $31^\circ\ 10'$.

Therefore, $90 - \theta = 31^\circ\ 10'$.

The value of θ can be found by solving the above equation for θ.

Then $\theta = 90 - 31^\circ\ 10' = \underline{\underline{58^\circ\ 50'}}$

Note: The cosine value of an angle in a triangle, like the sine value of an angle in a triangle, always takes a value from 0 to 1. Apart from the above method, the cosine value of an angle in a triangle can also be found directly from the Sines Table. Observe that in the Sines Table, just before the Mean Differences column, there is a column with angle values which are obtained by subtracting the angle values in the first column from 90°. The cosine values of angles in a triangle can also be found by using the table values corresponding to the angle values in this column. However, in this case, the values in the Mean Differences column need to be subtracted instead of added.

Now let us consider how to use the table in relation to cosine values.

Let us find the value of cos $4^\circ\ 20'$ using the table.

80°	0.9848	0.9853	0.9858	0.9863	0.9868	0.9872	0.9877	9	0	1	1	2	2	3	3	4	4
81	.9877	.9881	.9886	.9890	.9894	.9899	.9903	8	0	1	1	2	2	3	3	3	4
82	.9903	.9907	.9911	.9914	.9918	.9922	.9925	7	0	1	1	2	2	2	3	3	3
83	.9925	.9929	.9932	.9936	.9939	.9942	.9945	6	0	1	1	1	2	2	2	3	3
84	.9945	.9948	.9951	.9954	.9957	.9959	.9962	5	0	1	1	1	1	2	2	2	3
85	0.9962	0.9964	0.9967	0.9969	0.9971	0.9974	0.9976	4									
86	.9976	.9978	.9980	.9981	.9983	.9985	.9986	3									
87	.9986	.9988	.9989	.9990	.9992	.9993	.9994	2									
88	.9994	.9995	.9996	.9997	.9997	.9998	.9998	1									
89	0.9998	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	0°									
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

(අන්තරය ඉතා කුඩා බැවින් වගු ගත කිරීම අනවශ්‍යය.)

ප්‍රකෘති කෝසයින
இயற்கைக் கோசைன்கள்
NATURAL COSINES

Example 4

We need to consider the row corresponding to 4° in the "degrees" column on the right hand side and the column corresponding to $20'$ in the "minutes" column at the bottom. The value in the table in the row containing 4°, (found to the left), and in the column containing $20'$ is 0.9971.

Therefore, $\cos 4^\circ\ 20' = 0.9971$.

Example 5

Now let us find the value of $\cos 9^\circ\ 26'$.

We see that $\cos 9^\circ\ 20' = 0.9868$ and that the value corresponding to $6'$ is 0.0003.

Now to obtain the value of $\cos 9^\circ\ 26'$, the value obtained from the Mean Differences column has to be subtracted from 0.9868.

Accordingly,

$$\cos 9^\circ\ 26' = 0.9868 - 0.0003$$
$$= \underline{\underline{0.9865}}$$

Example 6

Now let us find the angle θ such that $\cos \theta = 0.4374$.

25	0.4226	0.4253	0.4279	0.4305	0.4331	0.4358	0.4384	64	3	5	8	10	13	16	18	21	24
26	.4348	.4410	.4436	.4462	.4488	.4514	.4540	63	3	5	8	10	13	16	18	21	23
27	.4540	.4566	.4592	.4617	.4643	.4669	.4695	62	3	5	8	10	13	15	18	21	23
28	.4695	.4720	.4746	.4772	.4797	.4823	.4848	61	3	5	8	10	13	15	18	20	23
29	.4848	.4874	.4899	.4924	.4950	.4975	.5000	60°	3	5	8	10	13	15	18	20	23
⋮	⋮							⋮	⋮								
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

ප්‍රකෘති කෝසයින
இயற்கைக் கோசைன்கள்
NATURAL COSINES

The value in the table closest to 0.4374 and less than it is 0.4358. The angle which has this value as its cosine is $64^\circ\ 10'$ according to the table. The value 0.0016 which is the difference between 0.4374 and 0.4358 is found in the column corresponding to 6'. This needs to be subtracted from $64^\circ\ 10'$.

$64^\circ\ 10' - 6' = 64^\circ\ 4'$

$\therefore$ the angle θ such that $\cos\theta = 0.4374$ is $64^\circ\ 4'$.

Exercise 18.4

1. Find each of the following values using the Tangents Table.

a. $\tan 25^\circ$ **b.** $\tan 37^\circ$ **c.** $\tan 40^\circ\ 54'$

2. Find the angle θ corresponding to each of the tangent values given below.

a. $\tan\theta = 0.3214$ **b.** $\tan\theta = 0.7513$ **c.** $\tan\theta = 0.9432$

3. Find each of the following values using the Sines Table.

a. $\sin 10^\circ\ 30'$ **b.** $\sin 21^\circ\ 32'$ **c.** $\sin 25^\circ\ 57'$

4. Find the angle θ corresponding to each of the sine values given below.

a. $\sin\theta = 0.5000$ **b.** $\sin\theta = 0.4348$ **c.** $\sin\theta = 0.6437$

5. Find each of the following values using the Cosines Table. Examine the accuracy of your answers by using the Sines Table.

a. $\cos 5^\circ\ 40'$ **b.** $\cos 29^\circ\ 30'$ **c.** $\cos 44^\circ\ 10'$

6. Find the angle θ corresponding to each of the cosine values given below.

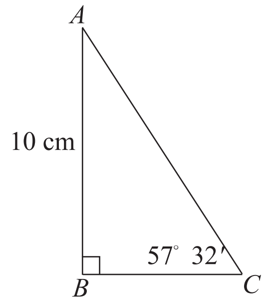
a. $\cos\theta = 0.4358$ **b.** $\cos\theta = 0.6450$ **c.** $\cos\theta = 0.9974$

18.5 Solving problems using the trigonometric tables

The types of problems that were solved earlier involving the angles 30°, 45° and 60° only can now be solved for any angle in a triangle. It is important to take the following into consideration when solving problems related to trigonometry.

1. Consider a suitable right angled triangle.
2. Select a suitable angle in the triangle.
3. Use a suitable trigonometric ratio corresponding to the selected angle.

Now let us consider some examples.



Example 1

Find the length of the side AC by using the information in the triangle ABC in the figure.

The angle in the triangle that is given is C. The length that is given is of the side opposite this angle. The length of the hypotenuse needs to be found. $\therefore$ The trigonometric ratio which involves these two sides, namely the sine ratio needs to be used.

$$\sin 57^\circ\ 32' = \frac{AB}{AC}$$

$$0.8437 = \frac{10}{AC}$$

$$\therefore \qquad AC = \frac{10}{0.8437}$$

Let us Find this value using the logarithms table.

$$\text{Let } AC = \frac{10}{0.8437}\ .$$

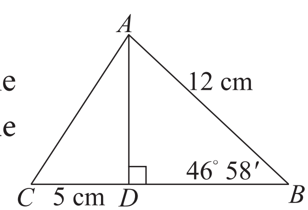
$$\text{Then,} \quad \log AC = \log \frac{10}{0.8437}$$

$$= \log 10 - \log 0.8437$$
$$= 1 - \bar{1}.9262$$
$$= 1.0738$$
$$\therefore AC = \text{antilog } 1.0738$$
$$\therefore AC = 11.85$$

Therefore the length of AC (accurate to the second decimal place) is 11.85 cm.

Example 2

AD has been drawn perpendicular to the side BC of the triangle ABC. Find the magnitude of $A\hat{C}B$ using the information in the figure.



Here, the right angled triangle that needs to be considered to find the angle $A\hat{C}B$ is the triangle ADC. If the lengths of two sides of this triangle are known, then the required angle can be found. The length of one of its side, CD has been given as 5 cm. We need to find the length of one more side. Note that we can find the length of AD by considering the triangle ABD. Therefore, let us first find the length of AD by considering the triangle ABD and using the sine ratio.

$$\sin\ 46^{\circ}\ 58' = \frac{AD}{AB}$$

$$0.7310 = \frac{AD}{12}$$

$$12 \times 0.7310 = AD$$

$$\therefore \quad AD = 8.7720 \text{ cm}$$

Now, considering the right triangle ACD, $\tan\ A\hat{C}D = \dfrac{AD}{CD}$

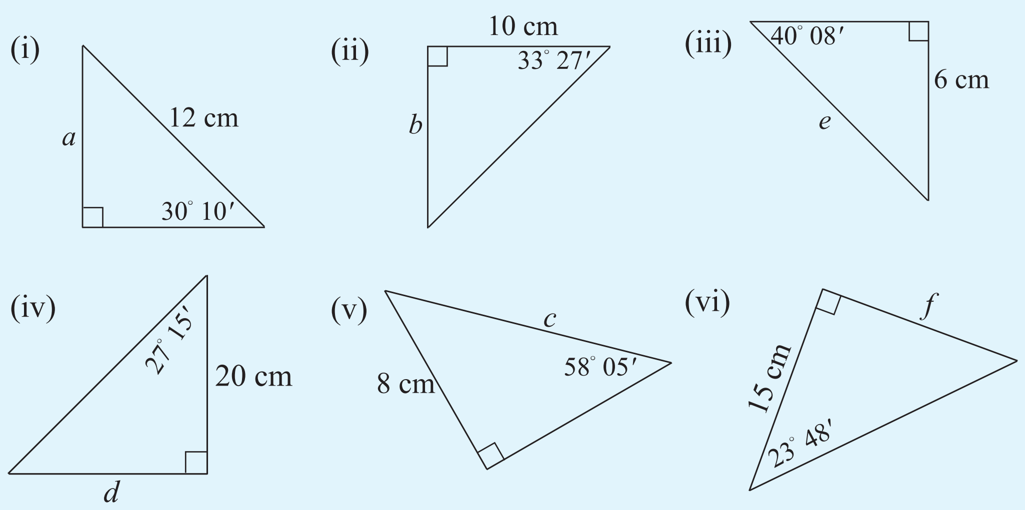
$$= \frac{8.7720}{5}$$

$$\therefore \quad \tan\ A\hat{C}D = 1.7544$$

$$\therefore \quad \underline{\underline{A\hat{C}D = 60^{\circ}\ 18'}}$$

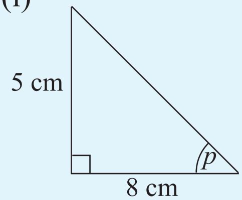
Exercise 18.5

1. Find the length denoted by an algebraic symbol in each of the following triangles.

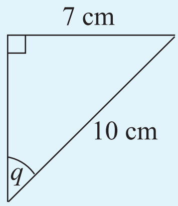


2. Find the angle denoted by an algebraic symbol in each of the following triangles.

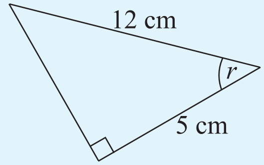
(i)



(ii)



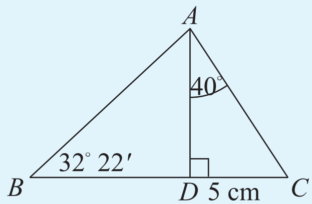
(iii)



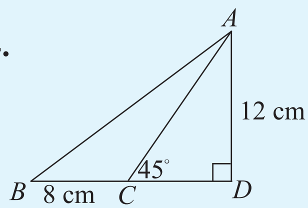
3. Based on the information in the given figure, find

(i) the perimeter of the triangle ABC.

(ii) the area of the triangle ABC.

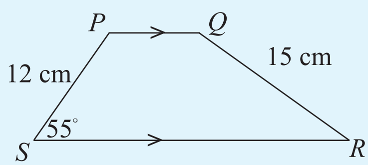


4.



Show using the information provided in the figure that the magnitude of the angle $A\hat{B}C$ of the triangle ABC, is $30^\circ\ 58'$.

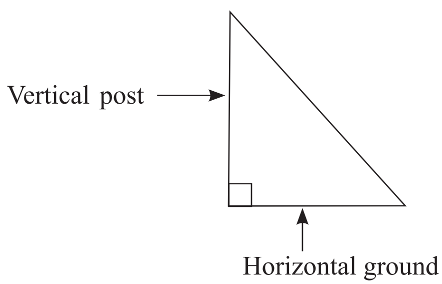
5.



In the trapezium $PQRS$, $SR > PQ$. If $PS = 12$ cm and $QR = 15$ cm, find the magnitude of $Q\hat{R}S$.

18.6 Angles in a vertical plane

A plane which is parallel to the earth (flat ground) is a horizontal plane. A plane which is perpendicular to a horizontal plane is a vertical plane. A post which is fixed perpendicular to the earth is a vertical post. Such a post is depicted in the figure.



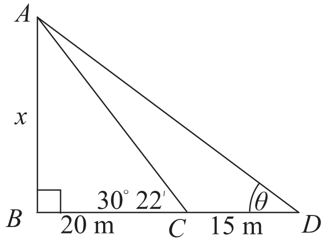
You learnt in grade 10 to determine locations using scale diagrams involving angles of elevation and angles of depression. Now let us learn how to find using the trigonometric ratios.

Let us consider the following example.

Example 1

A man is standing on the flat ground at a point C which is 20 m away from the foot of a vertical pillar AB. The angle of elevation of the top of the pillar from this point is $30^\circ\ 22'$. The man travels 15 m from this point along a straight path away from the pillar to another point, and again observes the top of the pillar.

(i) Represent this information in a rough sketch.
(ii) Find the height of the pillar to the nearest metre.
(iii) Find the angle of elevation of the top of the pillar from the second location.

(i) 

(ii) Let us take the height of the pillar to be x metres.

Then, considering the right angled triangle ABC we obtain,

$$\tan 30^\circ\ 22' = \frac{AB}{BC}$$

$$\tan 30^\circ\ 22' = \frac{x}{20}$$

$$\begin{aligned} x &= 20 \tan 30^\circ\ 22' \\ &= 20 \times 0.5859 \\ &= 11.718 \end{aligned}$$

Therefore, the height of the pillar is approximately 12 m.

(iii) Let us take the angle of elevation of the top of the pillar from D to be θ.
Then, considering the right angled triangle ABD we obtain,

$$\tan \theta = \frac{AB}{BD}$$

$$\tan\theta = \frac{12}{35}$$

$$\tan\theta = 0.3428$$

$$\therefore \quad \theta = 18^\circ\, 55'$$

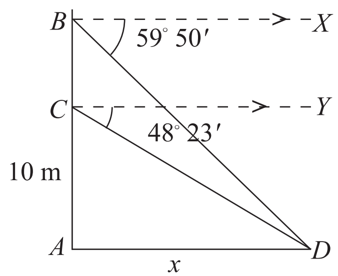
$\therefore$ the angle of elevation of the pillar from the second position is $18^\circ\, 55'$.

Example 2

A person stands by a small window of a vertical building consisting of several floors. The window is located at a height of 10 m above the flat ground. The person observes a motorcycle which is parked on the flat ground a long distance from the building. The angle of depression of the motorcycle from the window is $48^\circ\, 23'$. The person now ascends to the topmost storey of the building and observes the same motorcycle from another window. The angle of depression of the motorcycle when observed from this window is $59^\circ\, 50'$.

(i) Represent this information in a sketch.
(ii) How far from the building is the motorcycle parked?
(iii) Calculate the height of the window in the topmost storey of the building from the flat ground, to the nearest second decimal place.

(i)



(ii) ACD in the figure is a right angled triangle. Let us take the distance from the building to the motorcycle to be x m.

Since $Y\hat{C}D = 48^\circ\, 23'$, $A\hat{D}C = 48^\circ\, 23'$ (alternate angles)
Therefore, by considering the right angled triangle ADC.

$$\tan 48^\circ\, 23' = \frac{AC}{AD}$$

$$\tan 48^\circ\, 23' = \frac{10}{x}$$

$$\therefore \quad \frac{10}{\tan 48^\circ\, 23'} = x$$

Finding the value of x using the logarithms table

$\lg x = \lg 10 - \lg 1.1257$
$\quad = 1 - 0.0515$

$\therefore x = \text{antilog } 0.9485$
$\quad = 8.883$

That is, $x = \dfrac{10}{1.1257}$

$= 8.883$

$\therefore$ the distance from the building to the motorcycle is 8.883 m.

(iii) In the right angled triangle ABD, $A\hat{D}B = 59^\circ\ 50'$.

$$\tan 59^\circ\ 50' = \frac{AB}{AD}$$

$$\tan 59^\circ\ 50' = \frac{AB}{8.883}$$

$$AB = 8.883 \times 1.7205$$
$$= 15.28$$

$\therefore$ the window in the topmost storey of the building is located at a height of approximately 15.28 metres from the flat ground.

Do the following exercise according to the above examples.

Exercise 18.6

1. Draw a sketch based on the given information.

(i) A is located at the top of a vertical post AB. A person stands on the flat ground at a distance of 20 m from the foot of this post. The angle of elevation of the top of the pillar from the position of the eyes of the person is $55^\circ\ 20'$. The person is 1.5 m tall.

(ii) A technician attending to repairs, seated at the top of a vertical telecommunication pillar of height 35 m, observes a vehicle parked on the flat ground a long distance from the foot of the pillar. The angle of depression of the vehicle from the position of the technician is 50°.

(iii) A person on the second floor of a vertical building, observes a lighthouse at a distance of 75 m from the building. The angle of elevation of the top of the lighthouse from his position is $27^\circ\ 35'$ and the angle of depression of the foot of the lighthouse is $41^\circ\ 15'$.

(iv) The angle of elevation of the top of a utility post from the location where a child stands is 30°. The child travels a distance of 25 m along a straight path towards the utility post. The angle of elevation of the top of the utility post from this position is 50°. (Nelect the height of the child).

2. A security guard looking out from a window at the top of a lighthouse of height 20 m observes a ship travelling in the sea. The angle of depression of the ship from the top window is $30^\circ\ 15'$. Calculate the distance of the ship from the lighthouse.

3. The angle of elevation of the top of a vertical post from a point on the flat ground at a distance of 20 m from the foot of the post is $35^\circ\ 12'$. It is required to fix a taut wire from a point on the flat ground at a distance of 20 m from the foot of the post, to the top of the post to keep the post vertical. Find the length of the wire required for this. (Assume that half a meter of wire is used up to tie the wire)

4. The angle of elevation of the top of a vertical utility post fixed to the flat ground is 50° when observed from a point on the flat ground a certain distance from the foot of the post. If the height of the post is 12 m, find the distance from the foot of the post to the point of observation. (Neglect the height of the observer).

5. Two vertical posts A and B are fixed to the flat ground, a distance of 200 m from each other. The angle of elevation of the top of B, from the top of A is $4^\circ\ 10'$, and the angle of depression of the foot of B from this location is $8^\circ\ 15'$.

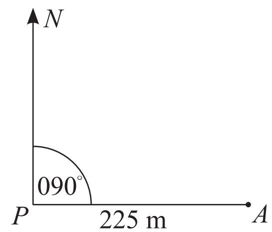
(i) Represent this information in a sketch.
(ii) Find the heights of the two posts A and B to the nearest metre.
(iii) Find the angle of elevation of the top of B from the foot of A.

6. A person stands right at the centre, between two vertical posts which are at a distance of 20 m from each other. The angle of elevation of the top of one post from this position is 60° while the angle of elevation of the top of the other post is 30° (neglect the height of the person).
(i) Find the heights of the two posts.
(ii) A taut wire has been drawn from the top of one post to the top of the other post. Find the length of the wire.

18.7 Angles in a horizontal plane

You have learnt earlier that bearings are used to indicate directions in a horizontal plane. Bearings provide a measure of an angle that is measured starting from the North and moving in a clockwise direction. Bearings are given using three digits. In modern measuring instruments the distance is also given with the bearing.

The point A which lies to the East of P, is located at a distance of 225 m from P on a bearing of 090°. This can be represented in a figure as follows.



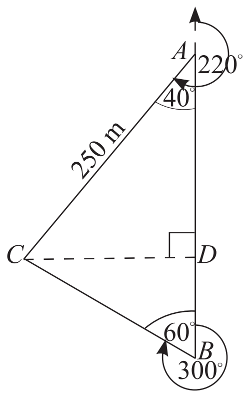
Let us see how calculations involving bearings are performed using trigonometric ratios by considering the following example.

Example 1

A straight road runs from the South to the North. When observed from a point A on this road, point C is located at a distance of 250 m on a bearing of 220°. When observed from another point B on the same straight road, C is located on a bearing of 300°.

(i) Represent the above information in a sketch.
(ii) Find the distance from C to the straight road.
(iii) Find the distance AB.

(i)



(ii) Since the bearing of C from A is 220°, $D\hat{A}C = 220^\circ - 180^\circ$
$= 40^\circ$

Then by considering the right triangle ACD we obtain, $\sin 40^\circ = \dfrac{CD}{AC}$.

$$AC \sin 40^\circ = CD$$
$$CD = 250 \sin 40^\circ$$
$$= 250 \times 0.6428$$
$$= 160.7000$$

$\therefore$ the shortest distance from C to the straight road AB is 160.7 m.

(iii) The length of $AB = AD + DB$

By considering the right triangle ACD, we obtain $\cos 40^\circ = \dfrac{AD}{AC}$

$$AD = AC\cos 40^\circ$$
$$= 250 \times 0.7660$$
$$= 191.5000$$
$$= 191.5 \text{ m}$$

By considering the right triangle BDC, we obtain $\tan 60^\circ = \dfrac{CD}{DB}$

$$DB = \frac{CD}{\tan 60^\circ}$$
$$= \frac{160.7}{1.732}$$
$$= 92.78 \text{ m}$$

$$\therefore \text{ the length of } AB = 191.5 + 92.78 \text{ m}$$
$$= \underline{\underline{284.28 \text{ m}}}$$

Exercise 18.7

1. Represent the following information in a sketch.
   (i) B is located at a distance of 12 m from A on a bearing of 080° .
   (ii) Q is located at a distance of 50 m from P on a bearing of 120°, and R is located at a distance of 25 m from Q on a bearing of 040°.
   (iii) Y is located at a distance of 30 m from X on a bearing of 150°, Z is located at a distance of 100 m from Y on a bearing of 200° and A is located at a distance of 50 m from Z on a bearing of 080°.

2. A motorcyclist who starts a journey from a location A, travels 8 km to the East and then turns to the North and travels a further 6 km till he reaches the location B.
   (i) Represent this information in a sketch.
   (ii) Find the bearing of A from B.
   (iii) Find the shortest distance between A and B.

3. A ship leaves harbour A and travels a distance of 150 km on a bearing of 040° until it reaches harbor B.
   (i) How far towards the North is harbor B from harbour A?
   (ii) How far towards the East is harbor B from harbour A?

4. A student who is trying to measure the breadth of a river which has straight parallel banks on the two sides, sits on one bank of the river and observes a tree, located directly in front of him on the opposite bank; the direction of the tree from the boy being perpendicular to the river banks. When the boy travels a distance of 75 m along the bank, he observes that the bearing of the tree from this location is 210°. Represent this information in a sketch and find the breadth of the river to the nearest metre using trigonometric ratios.

5. A group of forest conservationists observes from a distance that a fire has commenced in a forest. Using information they received on the location of the fire, starting from their camp *C*, they travel 2.5 km along a main road *A* on a bearing of 070° to the location P, and then from P by travelling 1.5 km on a bearing of 340°, they reach the location F of the fire.

(i) Represent this information in a figure.

(ii) Show with reasons that the group of conservationists was able to reach the location of the fire as quickly as possible, due to turning off the main road at *P*.

(iii) On what bearing would the conservationists have first observed the fire from their camp?

18.8 Using the calculator to find trigonometric ratios

When performing calculations involving trigonometric ratios using a scientific calculator, first the $\boxed{\text{MODE}}$ key should be used to display "DEG" on the screen.

Let us see how these calculations are performed by considering some examples.

Example 1

Express using a flowchart how the keys of a calculator need to be activated to obtain the following values.

(i) $\tan 35^\circ$ (ii) $\sin 35^\circ$ (iii) $\cos 35^\circ$

(i) $\tan 35^\circ$ $\boxed{\text{ON}}$ — $\boxed{\tan}$ — $\boxed{3}$ — $\boxed{5}$ — $\boxed{=}$ → $\boxed{0.7002}$

(ii) $\sin 35^\circ$ $\boxed{\text{ON}}$ — $\boxed{\sin}$ — $\boxed{3}$ — $\boxed{5}$ — $\boxed{=}$ → $\boxed{0.5736}$

(iii) $\cos 35^\circ$ $\boxed{\text{ON}}$ — $\boxed{\cos}$ — $\boxed{3}$ — $\boxed{5}$ — $\boxed{=}$ → $\boxed{0.8192}$

Example 2

Calculate the value of θ in each of the following cases.

(i) $\tan \theta = 1.2131$ (ii) $\sin \theta = 0.7509$ (iii) $\cos \theta = 0.5948$

(i) $\boxed{\text{ON}}$ — $\boxed{\text{SHIFT}}$ — $\boxed{\tan}$ — $\boxed{1}$ — $\boxed{.}$ — $\boxed{2}$ — $\boxed{1}$ — $\boxed{3}$ — $\boxed{1}$ — $\boxed{=}$ → $\boxed{50.5^\circ}$

(ii) $\boxed{\text{ON}}$ — $\boxed{\text{SHIFT}}$ — $\boxed{\sin}$ — $\boxed{0}$ — $\boxed{.}$ — $\boxed{7}$ — $\boxed{5}$ — $\boxed{0}$ — $\boxed{9}$ — $\boxed{=}$ → $\boxed{48.66^\circ}$

(iii) $\boxed{\text{ON}}$ — $\boxed{\text{SHIFT}}$ — $\boxed{\cos}$ — $\boxed{0}$ — $\boxed{.}$ — $\boxed{5}$ — $\boxed{9}$ — $\boxed{4}$ — $\boxed{8}$ — $\boxed{=}$ → $\boxed{53.5^\circ}$

Note: Observe that the value of each angle is obtained only in degrees.
For example $50.5^\circ = 50^\circ\,30'$

Exercise 18.8

1. Write down the order in which the keys of a calculator need to be activated to obtain the (i) tan value (ii) sin value (iii) cos value of the following angles.

a. 40° **b.** 75° **c.** 88° **d.** 43°

2. Express using a flowchart how the keys of a calculator need to be activated to obtain the value of θ in each of the following cases.

a. $\sin\theta = 0.9100$ **d.** $\cos\theta = 0.1853$ **g.** $\tan\theta = 0.5736$
b. $\sin\theta = 0.7112$ **e.** $\cos\theta = 0.7089$ **h.** $\tan\theta = 0.7716$
c. $\sin\theta = 0.1851$ **f.** $\cos\theta = 0.4550$ **i.** $\tan\theta = 0.9827$

Miscellaneous Exercise

1. Two ships P and Q leave a harbor simultaneously. Both ships travel at the same uniform speed of 18 kilometres per hour. P travels from the harbor on a bearing of 010° while Q travels on a bearing of 320°. Find the distance between the two ships after an hour.

2. Two tall buildings are located on opposite sides of a road. One building is 9 m taller than the other. The angle of elevation of the top of the shorter building from the foot of the taller building is $42^\circ\,20'$ If the shorter building is of height 15 m, determine the following. (Neglect the height of the observer)

(i) The distance between the two buildings.
(ii) The angle of elevation of the top of the taller building from the foot of the shorter building.

3. $AB = 10$ cm, $BC = 7$cm and $A\hat{B}C = 30^\circ\,26'$. The perpendicular drawn from A to BC is AX. Find the area of the triangle ABC.

4. Two flagpoles have been fixed on flat ground. Two points A and B are located along the straight line joining the feet of the two flagpoles. The angles of elevation of the two flagpoles from A are 30° and 60° , and from B are 60° and 45°. The distance between A and B is 10 m.

(i) Find the heights of the two flagpoles.
(ii) Find the distance between the two flagpoles.

19 Matrices

By studying this lesson you will be able to,

- identify a matrix
- identify the elements and the order of a matrix
- add and subtract matrices
- multiply a matrix by an integer
- multiply two matrices
- solve problems related to matrices.

19.1 Introducing Matrices

The idea of matrices was introduced in 1854 by the British mathematician Arthur Cayley. Let us identify matrices using a simple example.

The marks obtained by Wimal, Farook and Radha in a term test, for the subjects Mathematics and Science are shown in the table below.

	Mathematics	Science
Wimal	75	66
Farook	72	70
Radha	63	81

The numbers in the table given above can be represented in a matrix as follows.

$$\begin{pmatrix} 75 & 66 \\ 72 & 70 \\ 63 & 81 \end{pmatrix}$$

Here the columns indicate the subjects while the rows indicate the students. This information can also be represented in a matrix as follows.

$$\begin{pmatrix} 75 & 72 & 63 \\ 66 & 70 & 81 \end{pmatrix}$$

Here the columns indicate the students while the rows indicate the subjects.

An array of numbers organized in rows and columns is known as a matrix. Given below are several examples of matrices.

(i) $\begin{pmatrix} 3 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & 2 & 1 \end{pmatrix}$ (iii) $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

(iv) $\begin{pmatrix} -2 & 3 \\ 2 & 0 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 4 & 3 \\ 2 & 6 & 1 \\ 5 & 2 & 2 \end{pmatrix}$ (vi) $\begin{pmatrix} -5 & 4 \\ 9 & -1 \\ 0 & 4 \end{pmatrix}$

The numbers in a matrix are called the elements of the matrix. The elements of a matrix, apart from being numbers may also be algebraic symbols or expressions which stand for numbers.

Matrices are named using capital letters of the English alphabet. In instances when elements are expressed in terms of algebraic symbols, simple letters of the English alphabet are used.

Example 1

Three matrices which are named are shown below.

$A = \begin{pmatrix} 5 & 4 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$ $B = \begin{pmatrix} 3 \\ 0 \\ 5 \end{pmatrix}$ $P = \begin{pmatrix} 2 & c \\ a & b \end{pmatrix}$

Example 2

In a Cartesian plane, the coordinates of the points A and B are (0, 5) and (4, 3) respectively. Represent this information in a matrix. Name it P.

In a table,

	A	B
x	0	4
y	5	3

As a matrix,

$P = \begin{pmatrix} 0 & 4 \\ 5 & 3 \end{pmatrix}$

Order of a Matrix

Consider the matrix $A = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 2 & 0 \end{pmatrix}$

The number of rows in A is 2 and the number of columns is 3. Using the number of rows and number of columns, we write the order of the matrix as 2×3. A is known as a "two by three" matrix.

Accordingly, A is sometimes written as,

$$A = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 2 & 0 \end{pmatrix}_{2 \times 3.}$$

Example 1

Write the order of each of the matrices given below.

(i) $\begin{pmatrix} 3 & 2 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}$

Number of rows in the matrix $= 3$
Number of columns in the matrix $= 2$
Order of the matrix $= 3 \times 2$

(ii) $\begin{pmatrix} 3 & 2 & 4 \end{pmatrix}$

Number of rows $= 1$
Number of columns $= 3$
Order of the matrix $= 1 \times 3$

(iii) $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Number of rows $= 2$
Number of columns $= 1$
Order of the matrix $= 2 \times 1$

(iv) $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Number of rows $= 2$
Number of columns $= 2$
Order of the matrix $= 2 \times 2$

Row Matrices, Column Matrices and Square Matrices

Matrices with only one row are known as **row matrices**, matrices with only one column are known as **column matrices** and matrices which have an equal number of rows and columns are known as **square matrices**. Since the number of columns and the number of rows of a square matrix are equal, the order of a square matrix with two rows and two columns is known as a square matrix of order 2 and a matrix with three rows and three columns is known as a square matrix of order 3 etc.

For example,

$A = \begin{pmatrix} 3 & 2 & 5 \end{pmatrix}$ is a row matrix.

$B = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ is a column matrix.

$C = \begin{pmatrix} 5 & 2 & 1 \\ 3 & 1 & 3 \\ 0 & 4 & 0 \end{pmatrix}$ is a square matrix.

Identity Matrices and Symmetric Matrices

$$P = \begin{pmatrix} 3 & 2 & 4 \\ 6 & 5 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

In the square matrix given above, the main diagonal is highlighted. The string of elements from the top leftmost corner to the bottom rightmost corner is the main diagonal.

Note: The main diagonal is defined only for square matrices. Most often, the main diagonal is known simply as the diagonal.

The main diagonal of a square matrix of order two is highlighted below.

$$Q = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$$

Given below is a special square matrix.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The main diagonal of A consists of only the number 1. Apart from the diagonal elements, all the other elements are 0. This type of matrix is known as an **identity matrix**. The matrix A is the identity matrix of order 3 × 3. Given below is the identity matrix of order 2 × 2.

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

When naming identity matrices, the letter I is used. The identity matrix with n rows and n columns is written as $I_{n \times n}$. Accordingly,

$$I_{3 \times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_{2 \times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Can you identify the special feature in the matrix given below?

$$X = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 0 & 4 \\ 1 & 4 & 5 \end{pmatrix}$$

Consider the elements around the main diagonal of X. The elements around the main diagonal which are equal to each other are placed symmetrically about the main diagonal. Such matrices are called **symmetric matrices**.

$$Y = \begin{pmatrix} 1 & 5 \\ 5 & 3 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The elements around the main diagonals of Y and Z which are equal are placed symmetrically about the main diagonal. Therefore these are symmetric matrices too.

Note: Symmetric matrices are defined only for square matrices.

Exercise 19.1

1. Sarath bought 2 oranges and 3 mangoes, Kamal bought 4 oranges and 1 mango and Raju bought 1 orange and 5 mangoes from a certain fruit stall.

(i) Express the amount of fruits bought by Sarath as a row matrix.
(ii) Express the amount of fruits bought by Kamal as a row matrix.
(iii) Express the amount of fruits bought by Raju as a row matrix.
(iv) Construct a matrix with quantities of fruits Sarath, Kamal and Raju bought respectively as its rows.

2. Write the order of each of the following matrices.

(i) $A = \begin{pmatrix} 3 & 2 \\ 5 & 1 \\ 4 & 3 \end{pmatrix}$ (ii) $B = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 2 & 2 \end{pmatrix}$ (iii) $C = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$

(iv) $D = \begin{pmatrix} 0 & 4 \end{pmatrix}$ (v) $E = \begin{pmatrix} 5 & 8 & 3 \end{pmatrix}$ (vi) $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

3. From the matrices given below, select the row matrices and the column matrices.

(i) $P = \begin{pmatrix} 3 & 0 & 2 \end{pmatrix}$ (ii) $Q = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (iii) $R = \begin{pmatrix} 4 & 3 \end{pmatrix}$

(iv) $S = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ (v) $T = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ (vi) $U = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}$

4. From the following matrices, select and write down the

(i) square matrices
(ii) symmetric matrices
(iii) identity matrices

Highlight the diagonals of the square matrices.

$A = \begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 2 & 1 \\ 4 & 0 & 4 \\ 2 & 2 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $F = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $G = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

19.2 Addition and Subtraction of Matrices

We have learnt how to add, subtract and multiply numbers. We have seen how easily we can solve problems by using these mathematical operations. We can similarly define mathematical operations on matrices too. Let us first consider how to add matrices.

Consider the two matrices A and B given below.

$$A = \begin{pmatrix} 4 & 1 \\ 2 & 0 \\ 9 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 6 \\ 3 & 9 \\ 2 & 8 \end{pmatrix}$$

Both these matrices have the same order, 3 × 2. The addition of A and B is defined as the matrix which is obtained when the corresponding elements of A and B are added together.

Accordingly,

$$A + B = \begin{pmatrix} 4 & 1 \\ 2 & 0 \\ 9 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 6 \\ 3 & 9 \\ 2 & 8 \end{pmatrix} = \begin{pmatrix} 5 & 7 \\ 5 & 9 \\ 11 & 13 \end{pmatrix}$$

By corresponding elements we mean the elements which are in the same position of the matrix. For example, the element in the first row and second column of the matrix A is 1. The corresponding element in matrix B is 6; that is the number in the first row and second column of B.

Now let us consider an example with algebraic symbols.

If $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ and $Y = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}$ then, $X + Y = \begin{pmatrix} x_1 + y_1 & x_2 + y_2 \\ x_3 + y_3 & x_4 + y_4 \end{pmatrix}$

The addition of matrices is defined only for matrices of the same order. Accordingly addition of matrices of different orders is not defined.
Now let us consider how the addition of two matrices is used through an example. Even though this example is a very simple one, it shows clearly how matrices can be used in practical situations.

Example 1

Praveen and Tharindu are two bowlers in the school cricket team. The number of wickets they took in the years 2014 and 2015 in one day and two day matches are shown in the tables given below.

	2014	2015
Praveen	21	23
Tharindu	15	16

Wickets taken in one day matches

	2014	2015
Praveen	14	16
Tharindu	9	19

Wickets taken in two day matches

Let us name the matrix which has the information on the one day matches as A and the one with the information on the two day matches as B.

Then we can write $A = \begin{pmatrix} 21 & 23 \\ 15 & 16 \end{pmatrix}$ and $B = \begin{pmatrix} 14 & 16 \\ 9 & 19 \end{pmatrix}$

In these matrices the columns represent years while the rows represent wickets. Let us find $A + B$.

$$A + B = \begin{pmatrix} 35 & 39 \\ 24 & 35 \end{pmatrix}$$

Think about what the matrix $A + B$ represents. It shows the total number of wickets Praveen and Tharindu took in 2014 and 2015 in both one day and two day matches. It can be represented in a table as shown below.

	2014	2015
Praveen	35	39
Tharindu	24	35

Total wickets taken

The subtraction of one matrix from another is also defined similarly. Here, the corresponding numbers are subtracted. For subtraction too, the orders of the two matrices should be equal. As an example,

$$A = \begin{pmatrix} 5 & 9 \\ 2 & 3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 4 \\ 6 & 0 \end{pmatrix} \text{ then, } A - B = \begin{pmatrix} 4 & 5 \\ -4 & 3 \end{pmatrix}.$$

Let us consider another example.

If X is the 3×3 matrix with all the elements equal to 2 and Y is the 3×3 identity matrix, then find the matrix $X - Y$.

$$X = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore,

$$X - Y = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

Equality of two matrices

Let us consider what it means when we say that two matrices are equal.

$$A = \begin{pmatrix} 2 & 3 \\ 10 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

For A and B to be equal matrices $a = 2$, $b = 3$, $c = 10$ and $d = 9$.
That is, for two matrices to be equal, their orders should be equal and the corresponding elements of the two matrices should be equal.

Exercise 19.2

1. Simplify the following matrices.

(i) $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$

(ii) $\begin{pmatrix} 3 & -2 & 3 \end{pmatrix} + \begin{pmatrix} 2 & -2 & -4 \end{pmatrix}$

(iii) $\begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 2 \\ -4 \end{pmatrix}$

(iv) $\begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$

(v) $\begin{pmatrix} 2 & -2 \\ 3 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 3 & -1 \\ -2 & 1 \\ 2 & -2 \end{pmatrix}$

(vi) $\begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}$

(vii) $\begin{pmatrix} 2 & 5 & -1 \\ 3 & 4 & 6 \\ 2 & 4 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -4 & 4 \\ -4 & 0 & 1 \\ 1 & 3 & 0 \end{pmatrix}$

(viii) $\begin{pmatrix} 5 & 4 & 2 \\ 2 & 3 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 5 \\ 5 & 3 & 10 \end{pmatrix}$

2. Simplify the following matrices.

(i) $\begin{pmatrix} 4 & 3 \\ 2 & 5 \\ 6 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & 2 \\ 2 & 1 \end{pmatrix}$

(ii) $\begin{pmatrix} -3 \\ 5 \\ 8 \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \\ 2 \end{pmatrix}$

(iii) $\begin{pmatrix} 3 & 2 \\ -1 & 4 \end{pmatrix} - \begin{pmatrix} -2 & 3 \\ 5 & 2 \end{pmatrix}$

(iv) $\begin{pmatrix} 5 & -3 & -2 \end{pmatrix} - \begin{pmatrix} 2 & -4 & -2 \end{pmatrix}$

(v) $\begin{pmatrix} 5 & 3 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$ (vi) $\begin{pmatrix} 6 & 3 & 2 \\ 4 & 3 & 5 \\ 3 & -2 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 2 & -2 \\ 2 & 0 & 2 \\ 1 & -5 & -4 \end{pmatrix}$

3. If $\begin{pmatrix} 2 & 3 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} a & b & c \end{pmatrix}$ find the values of a, b and c.

4. If $\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 5 & 4 \\ 3 & 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find the values of a, b, c and d.

5. If $\begin{pmatrix} 5 & 3 & 2 \\ 4 & 1 & 2 \end{pmatrix} + \begin{pmatrix} x & 2 & -1 \\ y & 1 & z \end{pmatrix} = \begin{pmatrix} 8 & 5 & 1 \\ 2 & 2 & 3 \end{pmatrix}$ find the values of x, y and z.

6. If $\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} x & 3 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ find the values of x and y.

19.3 Multiplication of a matrix by a number

Now let us consider what it means to multiply a matrix by a number. Multiplying a matrix by a number means multiplying each element in the matrix by that number. The matrix obtained by multiplying matrix A by the number k is written as kA. Let us only consider multiplying by an integer. For example, let

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & 8 & 1 \end{pmatrix}$$

When A is multiplied by 5 the matrix obtained is,

$$5A = \begin{pmatrix} 5 \times 3 & 5 \times 1 & 5 \times 0 \\ 5 \times (-2) & 5 \times 8 & 5 \times 1 \end{pmatrix} = \begin{pmatrix} 15 & 5 & 0 \\ -10 & 40 & 5 \end{pmatrix}.$$

When A is multiplied by -3 the matrix obtained is,

$$-3A = \begin{pmatrix} -3 \times 3 & -3 \times 1 & -3 \times 0 \\ -3 \times -2 & -3 \times 8 & -3 \times 1 \end{pmatrix} = \begin{pmatrix} -9 & -3 & 0 \\ 6 & -24 & -3 \end{pmatrix}.$$

Note: The matrix obtained by multiplying matrix A by a number k has the same order as A.

Example: If $X = \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 5 & -2 \\ 2 & 1 \end{pmatrix}$, then find the matrix $3X - 2Y$.

$$3X - 2Y = 3\begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix} + (-2)\begin{pmatrix} 5 & -2 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 12 \\ 3 & 0 \end{pmatrix} + \begin{pmatrix} -10 & 4 \\ -4 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 16 \\ -1 & -2 \end{pmatrix}$$

Exercise 19.3

1. Simplify the following matrices.

(i) $3\begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$ (ii) $4\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ (iii) $3\begin{pmatrix} 2 & -1 & 3 \\ -3 & 1 & 2 \end{pmatrix}$

(iv) $2\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ (v) $3\begin{pmatrix} 1 & 0 & -2 \\ 3 & -1 & 2 \\ -3 & 2 & 0 \end{pmatrix}$ (vi) $-2\begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix}$

2. Find a, b, c and d if $3\begin{pmatrix} 4 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

3. Find the values of x, y and z if $4\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8 \\ -12 \\ 2 \end{pmatrix}$

4. Find the values of x, y, a and b if $2\begin{pmatrix} 5 & x \\ -2 & 9 \end{pmatrix} - 3\begin{pmatrix} y & -5 \\ 4 & a \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ b & 0 \end{pmatrix}$

19.4 Multiplication of Matrices

From the above definitions you may have realized that addition and subtraction of matrices as well as multiplication of a matrix by a number are carried out as for numbers. However, multiplication of two matrices is done in a different way. This is shown below.

Initially let us look at how to multiply a row matrix and a column matrix. If A is a row matrix of order $1 \times m$ and B is a column matrix of order $m \times 1$, then the matrix AB is defined and is of order 1×1 . To describe how matrix multiplication is done in this case, let us consider an example.

Let us assume that $A = \begin{pmatrix} a_1 & a_2 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

Accordingly, the order of A is 1×2 while the order of B is 2×1. Then AB is defined as,

$$AB = (a_1b_1 + a_2b_2)_{\,1 \times 1}$$

Example 1

Find AB if $A = \begin{pmatrix} 5 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$$AB = (5 \times 3 + 2 \times 1) = (17)$$

We learnt earlier that any matrix can be multiplied by a number. We also learnt that addition and subtraction of matrices however can be done only if the matrices are of the same order. Multiplication of matrices too can be done in some cases only. Above we saw how to multiply a row matrix and a column matrix. We can multiply matrices which are of different orders too. In general, AB is defined if the order of A is $m \times n$ and the order of B is $n \times p$; that is, if the number of columns of A is equal to the number of rows of B. In this case we get a matrix of order $m \times p$, which is the number of rows of A into the number of columns of B. Let us now consider how such products of matrices are found.

As an example, let us see how to find AB if $A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}_{2 \times 2}$ and $B = \begin{pmatrix} 1 & 8 \\ 6 & 7 \end{pmatrix}_{2 \times 2}$

When multiplying the above two matrices, multiply each row of A by each column of B, in the same way that we multiplied a row matrix by a column matrix.

$$= \begin{pmatrix} (2\ \ 4)\begin{pmatrix} 1 \\ 6 \end{pmatrix} & (2\ \ 4)\begin{pmatrix} 8 \\ 7 \end{pmatrix} \\ (3\ \ 5)\begin{pmatrix} 1 \\ 6 \end{pmatrix} & (3\ \ 5)\begin{pmatrix} 8 \\ 7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 2 \times 1 + 4 \times 6 & 2 \times 8 + 4 \times 7 \\ 3 \times 1 + 5 \times 6 & 3 \times 8 + 5 \times 7 \end{pmatrix}$$

$$= \begin{pmatrix} 26 & 44 \\ 33 & 59 \end{pmatrix} \text{ (by finding each product)}$$

The way the above matrix AB was obtained can be explained as below.

- The element in the first row and first column of AB is obtained by multiplying the first row of A (row matrix) by the first column of B (column matrix).
- The element in the first row and second column of AB is obtained by multiplying the first row of A (row matrix) by the second column of B (column matrix).
- The element in the second row and first column of AB is obtained by multiplying the second row of A (row matrix) by the first column of B (column matrix).
- The element in the second row and second column of AB is obtained by multiplying the second row of A (row matrix) by the second column of B (column matrix).

Any two matrices for which the product is defined (that is, the number of columns of A is equal to the number of rows of B), can be multiplied as above. Let us look at a few more examples.

Example 2

If $X = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}$ and $Y = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$ then show that XY is defined and find it. Is YX defined?

Number of columns in $X = 2$ and the number of rows in $Y = 2$. As the number of columns in X and the number of rows in Y are equal, XY is defined.

Now,

$$XY = \begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 7 \end{pmatrix}$$

By multiplying each row of X by each column of Y we obtain,

$$= \begin{pmatrix} (4 \;\; 6)\begin{pmatrix} 1 \\ 7 \end{pmatrix} \\ (2 \;\; 3)\begin{pmatrix} 1 \\ 7 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 4 \times 1 + 6 \times 7 \\ 2 \times 1 + 3 \times 7 \end{pmatrix}$$

$$= \begin{pmatrix} 46 \\ 23 \end{pmatrix}$$

Now let us see whether YX is defined.
The number of columns in Y is 1 while the number of rows in X is 2. As the number of the columns in Y is not equal to the number of rows in X, YX is not defined.

Let $P = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 6 & 3 \end{pmatrix}$. Under this section on matrix multiplication we first defined the product of matrices of the form QP. This product can be found by using the above definition too. That is, by multiplying each row in Q by each column in P.

$$QP = \begin{pmatrix} 6 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = (9).$$

This is a matrix with just one element. A matrix with only one element can be considered as a number. Therefore we write $QP = 9$.

Furthermore PQ is also defined. The product of the matrices P and Q, namely PQ, is a 2×2 matrix.

$$PQ = \begin{pmatrix} 2 \\ -1 \end{pmatrix}\begin{pmatrix} 6 & 3 \end{pmatrix} = \begin{pmatrix} 2 \times 6 & 2 \times 3 \\ (-1) \times 6 & (-1) \times 3 \end{pmatrix} = \begin{pmatrix} 12 & 6 \\ -6 & -3 \end{pmatrix}$$

Exercise 19.4

1. Simplify the following matrices.

(i) $\begin{pmatrix} 3 & 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

(ii) $\begin{pmatrix} 3 & 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

(iii) $\begin{pmatrix} 2 & -1 \\ 0 & -2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(iv) $\begin{pmatrix} 1 & 5 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}$

(v) $\begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

(vi) $\begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$

(vii) $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(viii) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$

(ix) $\begin{pmatrix} 2 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

(x) $\begin{pmatrix} 2 & -3 \end{pmatrix} \times \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$

2. Find a and b if $\begin{pmatrix} 2 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} a & b \end{pmatrix}$

3. A, B and C are three matrices. $A \times B = C$. Fill in the blanks in the following table.

Order of matrix A	Order of matrix B	Order of matrix C
1×2	2×1	....
2×2	.... $\times 1$	....
....$\times 2$	.... $\times 1$	1×1
.... $\times$	$1 \times$	2×2
.... $\times 1$	.... $\times 2$	$1 \times$ .....

4. If $P = \begin{pmatrix} 2 & -1 \end{pmatrix}$, $Q = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and $R = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
find,
(i) $P \times Q$
(ii) $P \times R$
(iii) $Q \times R$.

5. If $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$

(i) Find AB .
(ii) Find BA .
(iii) What is the relationship between AB and BA?

6. $C = \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix}$, $D = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$

(i) Find CD .
(ii) Find DC .

20 Inequalities

By studying this lesson you will be able to

- solve inequalities of the form $ax + b \gtreqless cx + d$ and represent the solutions on a number line,
- express problems related to day to day activities as inequalities and solve them.

Do the review exercise given below to recall what was learnt in grade 10 about solving inequalities of the form $ax + b \gtreqless c$

Review Exercise

1. Solve each of the inequalities given below.

a. $3x - 2 > 4$ **b.** $\frac{x}{2} + 5 \leq 7$ **c.** $5 - 2x > 11$

d. $\frac{x}{-2} + 3 \leq 5$ **e.** $\frac{5x}{6} + 4 \geq 14$ **f.** $3 - 2x \geq 9$

20.1 Solving inequalities of the form $ax + b \gtreqless cx + d$

Let us now consider how to solve inequalities of the form $ax + b \gtreqless cx + d$ algebraically and represent the solutions geometrically on a number line.

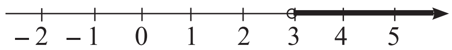
Example 1

Solve the inequality $3x - 2 > 2x + 1$ and represent the solutions on a number line.

When solving the inequality, all the terms with x should be carried to one side and the numerical terms should be carried to the other side (as in solving equations.)

$$3x - 2 > 2x + 1$$
$$3x - 2 + 2 > 2x + 1 + 2 \quad \text{(adding 2 to both sides)}$$
$$3x > 2x + 3$$
$$3x - 2x > 2x + 3 - 2x \quad \text{(subtracting } 2x \text{ from both sides.)}$$
$$\underline{\underline{x > 3}}$$

This is the solution of the inequality. In words we can say that the solutions of the inequality are all real numbers greater than 3. This can be represented on a number line as shown below.



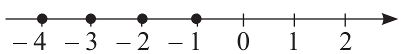
To represent the fact that 3 is not a solution, a small un-shaded circle is drawn around the point denoting 3.

Example 2

Solve the inequality $5x + 3 \leq 3x + 1$ and represent the solutions of x on a number line.

$$\begin{aligned}
5x + 3 &\leq 3x + 1 \\
5x + 3 - 3 &\leq 3x + 1 - 3 && \text{(subtracting 3 from both sides)} \\
5x &\leq 3x - 2 \\
5x - 3x &\leq 3x - 2 - 3x && \text{(subtracting } 3x \text{ from both sides)} \\
\frac{2x}{2} &\leq \frac{-2}{2} && \text{(dividing both sides by 2)} \\
\underline{\underline{x}} &\underline{\underline{\leq -1}}
\end{aligned}$$

Accordingly, the solutions of the inequality are all real numbers less than or equal to –1. The integral solutions of the inequality are all integers which are less than or equal to –1. That is –1, –2, –3 etc. This can be represented on a number line as shown below.



Note: If the integral solutions of the inequality are not asked for specifically, then all real numbers satisfying the inequality should be given as the solution.

Example 3

Solve the inequality $2x - 5 \geq 4x - 4$ and represent the solutions of x on a number line.

$$\begin{aligned}
2x - 5 &\geq 4x - 4 \\
2x - 5 + 5 &\geq 4x - 4 + 5 && \text{(adding 5 to both sides)} \\
2x &\geq 4x + 1 \\
2x - 4x &\geq 4x + 1 - 4x && \text{(subtracting } 4x \text{ from both sides)} \\
-2x &\geq 1 \\
\frac{-2x}{-2} &\leq -\frac{1}{2} && \text{(dividing both sides by –2)} \\
\underline{\underline{x}} &\underline{\underline{\leq \frac{-1}{2}}}
\end{aligned}$$

Note: Remember that when you divide by a negative number the inequality sign changes. Consider how to solve this problem without having to divide by a negative number.

Exercise 20.1

1. Solve each of the inequalities given below. Represent the integral solutions of each inequality on a number line.

a. $3x - 4 > 2x$ **b.** $6x + 5 \geq 5x$

c. $2x - 9 \leq 5x$ **d.** $8 - 3x > x$

e. $5 - 2x \leq 3x$ **f.** $12 - x > 3x$

2. Solve each of the inequalities given below, and for each inequality, represent all the solutions on a number line.

a. $2x - 4 > x + 3$ **b.** $3x + 5 < x + 1$

c. $3x + 8 \geq 3 - 2x$ **d.** $5x + 7 \geq x - 5$

e. $3x - 8 \leq 5x + 2$ **f.** $2x + 3 \geq 5x - 6$

g. $x - 9 > 6x + 1$ **h.** $5x - 12 \leq 9x + 4$

i. $\dfrac{3x + 2}{2} > x + 3$ **j.** $2x - 5 \leq \dfrac{3x - 4}{-2}$

20.2 Solving problems using inequalities

Example 1

Eight tea packets of the same mass and three 1kg packets of sugar are in a shopping bag. The maximum mass the bag can hold is 5kg.

(i) Taking the mass of one tea packet as x grammes, write an inequality in terms of x.

(ii) Solve the inequality and find the maximum mass that a tea packet could be.

It is easier to work this problem out if all the masses are converted to grammes.

(i) Mass of a tea packet in grammes $= x$
Mass of 8 tea packets in grammes $= 8x$
Mass of sugar in grammes $= 3 \times 1000$
$= 3000$
Maximum mass the bag can hold in grammes $= 5 \times 1000$
$= 5000$

According to the information given, $8x + 3000 \leq 5000$
This is the required inequality.

(ii) $8x + 3000 \leq 5000$
$8x + 3000 - 3000 \leq 5000 - 3000$
$\frac{8x}{8} \leq \frac{2000}{8}$
$x \leq 250$

$\therefore$ the maximum mass of a tea packet is 250g.

Example 2

Sarath bought 5 exercise books and 3 pens while Kamani bought 3 exercise books and 11 pens. The amount spent by Sarath was greater than or equal to the amount spent by Kamani. Moreover, the price of a pen was Rs 10.

(i) Taking the price of an exercise book as Rs x, write an inequality in terms of x.

(ii) By solving the inequality, find the minimum price of an exercise book.

(i) Price of the exercise books Sarath bought $=$ Rs $5x$
Amount Sarath spent $=$ Rs $5x + 30$
Similarly, the amount Kamani spent $=$ Rs $3x + 110$

According to the information given,

$$5x + 30 \geq 3x + 110$$

This is the required inequality.

(ii)
$$5x + 30 \geq 3x + 110$$
$$5x + 30 - 30 \geq 3x + 110 - 30$$
$$5x \geq 3x + 80$$
$$5x - 3x \geq 3x + 80 - 3x$$
$$\frac{2x}{2} \geq \frac{80}{2}$$
$$x \geq 40$$

$\therefore$ the minimum price of an exercise book is Rs 40.

Exercise 20.2

1. 5 bags of cement of mass 50kg each and 30 wires of equal mass have been loaded into a small tractor. The maximum mass the tractor can carry is 700kg.

(i) Taking the mass of a wire as x, construct an inequality using the given information.
(ii) Find the maximum mass of a wire.

2. There are twelve small packets of biscuits and five 200g packets of biscuits in box A, while in box B there are four small packets of biscuits and nine 200g packets of biscuits. The mass of the biscuits in box A is less than or equal to the mass of the biscuits in box B.

(i) Taking the mass of a small packet of biscuits as x, write an inequality in terms of x using the given information.
(ii) Find the maximum mass of a small packet of biscuits.

3. There are trained and untrained employees in a workplace. The daily wage of a trained employee is Rs 1200. The amount spent on the daily wages of 5 trained employees and 7 untrained employees is greater than or equal to the amount spent on the daily wages of 7 trained employees and 4 untrained employees.

(i) Taking the daily wage of an untrained employee to be Rs x, construct an inequality using the information given above.
(ii) Solve the inequality and find the minimum daily wage of an untrained employee.

4. 5 packets of tea of equal mass and 3 kg of sugar are packed in a polythene bag. The mass that the polythene bag is greater than or equal to the mass of 25 packets of tea. Using this information, construct an inequality and find the mass of a tea packet that the bag can hold.

5. Square tiles of two sizes are used to tile two rooms. The area of the larger tile is $900 cm^2$. To tile room A, 100 small tiles and 10 large tiles are needed, while to tile room B, 20 small tiles and 30 large tiles are needed. If the area of the floor of room B is greater than or equal to the area of the floor of room A, using an inequality, find the maximum length that a side of the smaller tile could be.

6. A large bucket of capacity 5 litres and a small bucket are used to fill a tank with water. The tank can be filled completely by using the large bucket 12 times and the small bucket 4 times (assuming that both buckets are filled to the brim). When the tank was filled using the large bucket 9 times and the small bucket 9 times, the tank did not overflow. Using an inequality, find the maximum capacity of the small bucket.

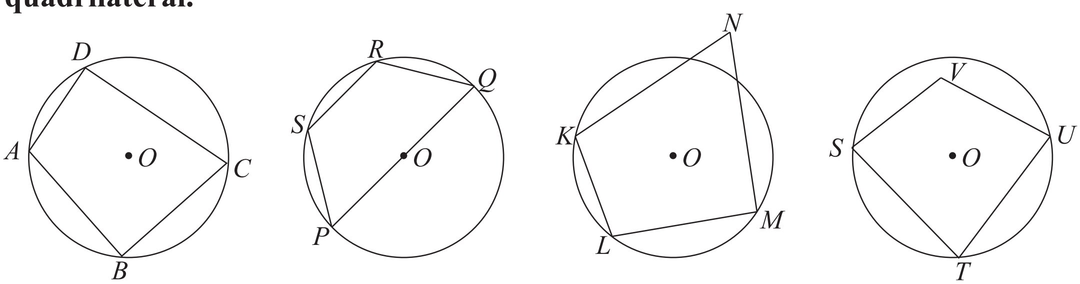
21 Cyclic Quadrilaterals

By studying this lesson you will be able to,

- identify cyclic quadrilaterals and identify the theorem that "the opposite angles of a cyclic quadrilateral are supplementary" and its converse,
- identify the theorem that "if one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral".

21.1 Cyclic quadrilaterals

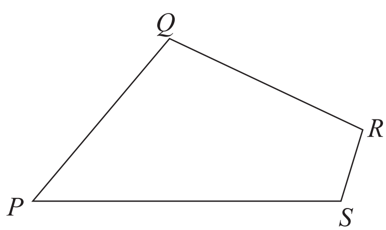
A quadrilateral which has all four vertices on the same circle is known as a **cyclic quadrilateral.**



(i) (ii) (iii) (iii)

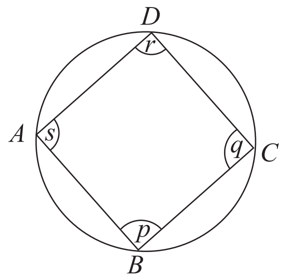
It is clear that the quadrilaterals in figures (i) and (ii) are **cyclic quadrilaterals** while the quadrilaterals in figures (iii) and (iv) are **not cyclic quadrilaterals**.

In a quadrilateral, the angle which is opposite a given angle is known as the "opposite angle". For example, in the quadrilateral shown below, $\hat{R}$ is the opposite angle of $\hat{P}$ and $\hat{S}$ is the opposite angle of $\hat{Q}$.



To understand the relationship between opposite angles in a cyclic quadrilateral let us do the following activity.

Activity



- Draw a cyclic quadrilateral as shown in the figure.
- Cut and separate out the angles in the cyclic quadrilateral.
- From the angles that were cut out, take the angles p and r and paste them on a piece of paper such that they are adjacent angles and see whether they form a pair of supplementary angles (that is, whether the sum of the magnitudes is 180°.) Do the same with the angles q and s.
- What is the conclusion you can draw regarding opposite angles of a cyclic quadrilateral?

You would have observed the $p + r = 180^\circ$ and $q + s = 180^\circ$. This relationship can be written as a theorem in the following form.

Theorem: The opposite angles of a cyclic quadrilateral are supplementary.

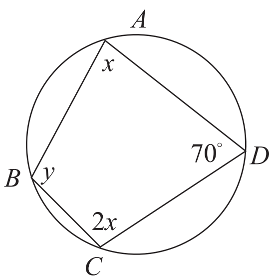
According to this theorem, in the above figure,

$$A\hat{B}C + C\hat{D}A = 180^\circ$$
$$D\hat{C}B + D\hat{A}B = 180^\circ$$

Let us now see how calculations are performed using the above theorem.

Example 1

Find the values of x and y in the cyclic quadrilateral $ABCD$ shown in the figure.



Since the opposite angles of a cyclic quadrilateral are supplementary,

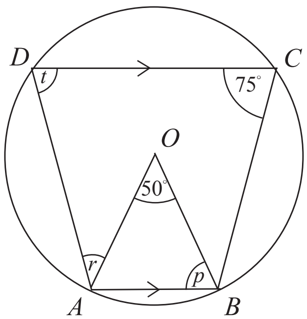
$$70^\circ + y = 180^\circ$$
$$\therefore \quad y = 180^\circ - 70^\circ$$
$$\underline{\underline{y = 110^\circ}}$$

Since the opposite angles of a cyclic quadrilateral are supplementary,

$$x + 2x = 180^\circ$$
$$3x = 180^\circ$$
$$\therefore \underline{\underline{x = 60^\circ}}$$

Example 2

O is the centre of the circle shown in the figure and *AB//DC*. Find the magnitudes of the angles denoted by symbols.



$\hat{OAB} = \hat{OBA}$ (*OA* and *OB* are equal as they are radii of the same circle)

$\therefore\ p + p + 50^\circ = 180^\circ$ (interior angles of the triangle *OAB*)

$$\therefore p = \frac{180^\circ - 50^\circ}{2}$$
$$= \underline{\underline{65^\circ}}$$

As the opposite angles of a cyclic quadrilateral add up to 180°,

$$75^\circ + D\hat{A}B = 180^\circ$$
$$D\hat{A}B = 180^\circ - 75^\circ$$
$$= 105^\circ$$

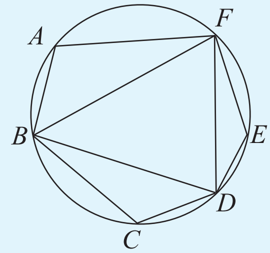
$$B\hat{A}O + O\hat{A}D = 105^\circ$$
$$\therefore\ 65^\circ + r = 105^\circ$$
$$r = 105^\circ - 65^\circ$$
$$r = \underline{\underline{40^\circ}}$$

As allied angles add up to 180°,

$$t + 105^\circ = 180^\circ$$
$$\therefore\ t = 180^\circ - 105^\circ$$
$$t = \underline{\underline{75^\circ}}$$

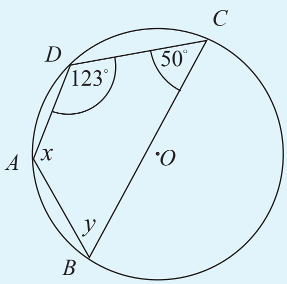
Exercise 21.1

1. (i) Write down all the cyclic quadrilaterals in the figure.
(ii) For each of the cyclic quadrilaterals written above, write down the pairs of opposite angles.

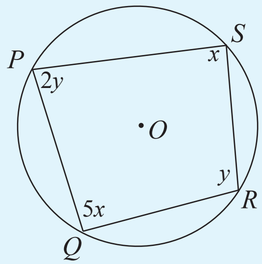


2. Find the magnitude of each of the angle denoted by a symbol, based on the information in the figure. O denotes the centre of each circle.

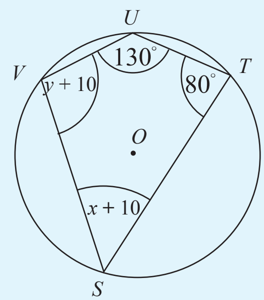
(i)



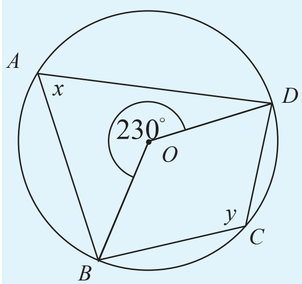
(ii)



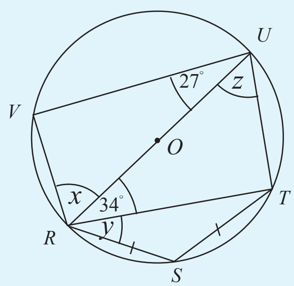
(iii)



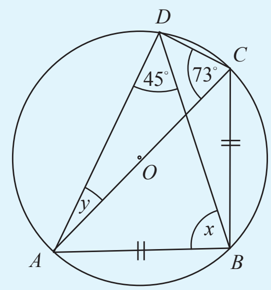
(iv)



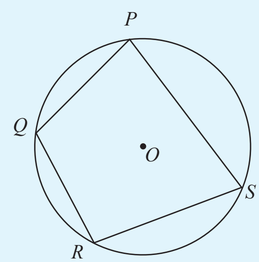
(v)



(vi)



3. O is the centre of the circle shown in the figure.



a. If $\hat{P} = 60^\circ$ and $\hat{S} = 125^\circ$, then find the magnitudes of $\hat{R}$ and $\hat{Q}$.

b. If $\hat{P} : \hat{R} = 2 : 3$, then find the magnitudes of $\hat{P}$ and $\hat{R}$.

c. If $\hat{Q} - \hat{S} = 120^\circ$, then find the magnitudes of $\hat{S}$ and $\hat{Q}$.

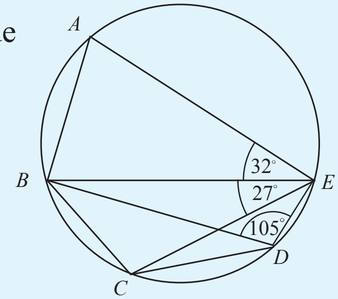
d. If $2\hat{P} = \hat{R}$, then find the magnitude of $\hat{P}$.

e. If $\hat{P} = 2x + y$, $\hat{Q} = x + y$, $\hat{R} = 60^\circ$ and $\hat{S} = 90^\circ$ then find the values of x and y.

4. The points denoted by A, B, C, D, E and F lie on the circumference of the circle with centre O. Find the value of $F\hat{A}B + B\hat{C}D + D\hat{E}F$.

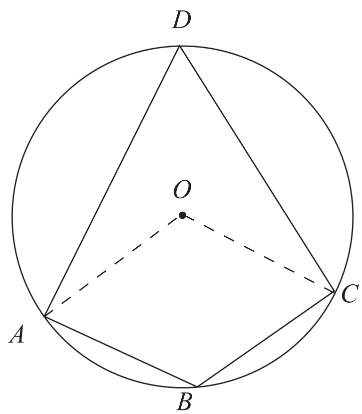
5. Using the information in the figure, find the magnitude of each of the angles given below.

a. $B\hat{A}E$ **b.** $C\hat{B}A$ **c.** $C\hat{B}E$



Now let us see how to prove the theorem, "the opposite angles of a cyclic quadrilateral are supplementary."

Data: $ABCD$ is a cyclic quadrilateral with its vertices on the circle with centre O.



To be proved: $A\hat{B}C + A\hat{D}C = 180^{\circ}$ and

$D\hat{A}B + D\hat{C}B = 180^{\circ}$.

Construction: Join OA and OC.

Proof: $A\hat{O}C = 2\,A\hat{D}C$
(the angle subtended at the centre is twice the angle subtended at the circumference)

$A\hat{O}C$ (reflex) $= 2\,A\hat{B}C$ (the angle subtended at the centre is twice the angle subtended at the circumference)

$\therefore$ $A\hat{O}C + A\hat{O}C$ (reflex) $= 2\,A\hat{D}C + 2\,A\hat{B}C$

But, $A\hat{O}C + A\hat{O}C$ (reflex) $= 360^{\circ}$ (angles around a point)

$\therefore$ $2\,A\hat{D}C + 2\,A\hat{B}C = 360^{\circ}$

Therefore, $A\hat{D}C + A\hat{B}C = 180^{\circ}$

We can join OB and OD, and similarly prove that $D\hat{A}B + D\hat{C}B = 180^{\circ}$.

$\therefore$ The opposite angles of a cyclic quadrilateral are supplementary.

The converse of this theorem is also true. That is, if the sum of the opposite angles of a quadrilateral is 180°, then its vertices lie on a circle. We can write this as a theorem as below.

Theorem: If the opposite angles of a quadrilateral are supplementary, then the vertices of the quadrilateral are on the circle.

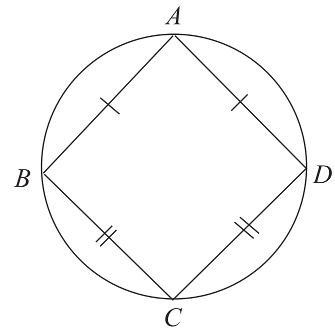
Now let us consider how riders are proved using the above theorem.

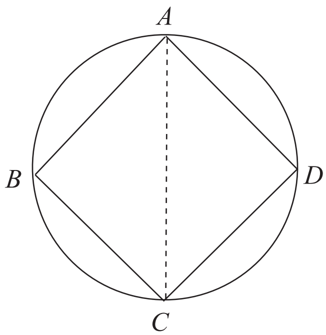
Example 1

In the cyclic quadrilateral shown in the figure, $AB = AD$ and $CB = CD$.

(i) Show that $\Delta ABC \equiv \Delta ACD$

(ii) Deduce that AC is a diameter.





(i) When we consider the triangles ABC and ADC,

$AB = AD$ (given)

$BC = DC$ (given)

AC is the common side

$\therefore \quad \Delta ABC \equiv \Delta ACD \quad$ (SSS)

(ii) $A\hat{B}C = A\hat{D}C$ (corresponding angles of congruent triangles are equal)

But, $A\hat{B}C + A\hat{D}C = 180^\circ$ (opposite angles of a cyclic quadrilateral are supplementary)

$\therefore \quad A\hat{B}C + A\hat{B}C = 180^\circ$

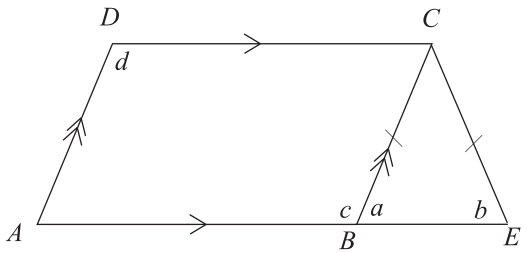
$\therefore \; 2\, A\hat{B}C = 180^\circ$

$\therefore \; A\hat{B}C = 90^\circ$

$\therefore \quad AC$ is a diameter (angle in a semicircle is 90°).

Example 2

In the parallelogram $ABCD$, AB is produced to E such that $CB = CE$. Show that $AECD$ is a cyclic quadrilateral.



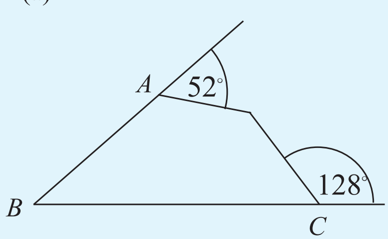
$a = b$ (since $CE = CB$)
$c = 180^\circ - a$ (angles on a straight line)
$c = 180^\circ - b$ (since $a = b$) ——— ①
$c = d$ (opposite angles of the parallelogram $ABCD$) ——— ②
From ① and ②,
$d = 180^\circ - b$
$\therefore\ b + d = 180^\circ$

As the opposite angles of the quadrilateral $AECD$ add up to 180°, this is a cyclic quadrilateral.

Exercise 21.2

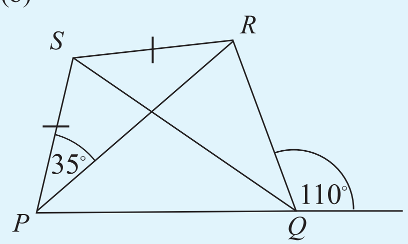
1. State with reasons whether each of the following quadrilaterals is a cyclic quadrilateral or not.

(a)



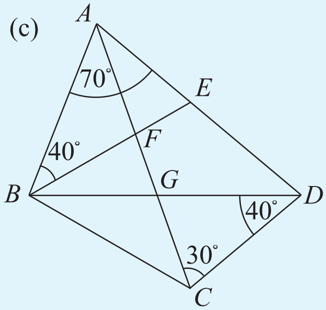
Quadrilateral $ABCD$

(b)



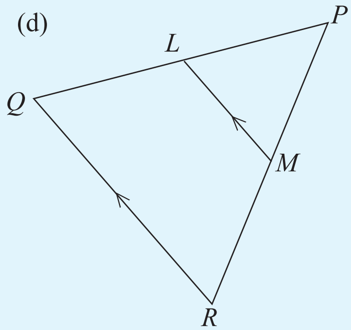
Quadrilateral $PQRS$

(c)



Quadrilateral $FGDE$

(d)



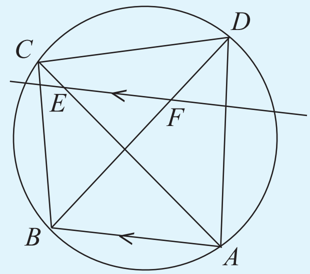
Quadrilateral $QRML$ if $PQ = PR$

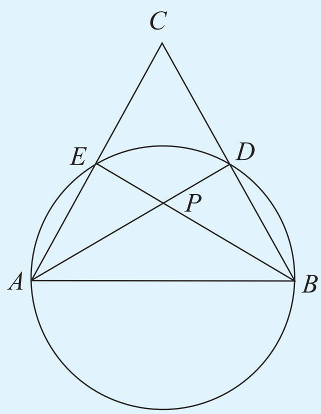
2. Show that $PQRS$ is a cyclic quadrilateral if $\hat{P} = \hat{Q}$ and $\hat{R} = \hat{S}$.

3. In the cyclic quadrilateral $ABCD$, AC is joined. Show that $B\hat{A}C = A\hat{D}C - A\hat{C}B$.

4. Show that A, B, C and D are points on the same circle if $A\hat{B}D + A\hat{D}B = D\hat{C}B$ in the quadrilateral $ABCD$.

5. Using the information in the figure, prove that $CDFE$ is a cyclic quadrilateral.





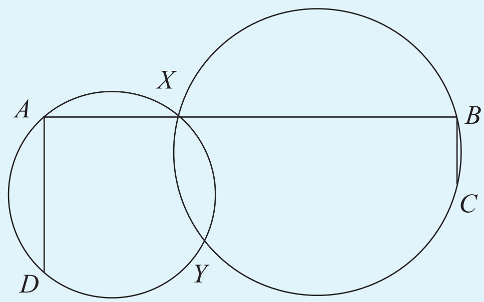
6. If AB is a diameter of the circle in the figure, show that $A\hat{P}B = C\hat{A}B + A\hat{B}C$.

7. In the triangle PQR, PQ is produced to S and PR is produced to T. The bisectors of $S\hat{Q}R$ and $Q\hat{R}T$ meet at X and the bisectors of $P\hat{Q}R$ and $P\hat{R}Q$ meet at Y.

(i) Show that $QXRY$ is a cyclic quadrilateral with XY as a diameter of the corresponding circle.

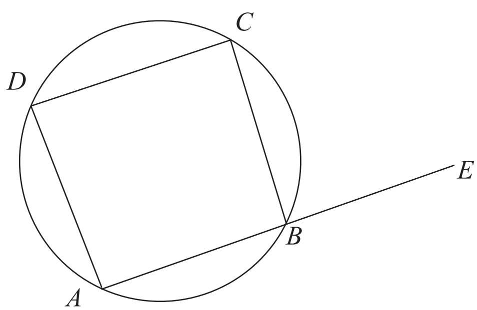
(ii) Find the magnitude of $Q\hat{X}R$ if $Q\hat{P}R = 40^\circ$.

8. The two circles in the figure intersect at X and Y. A straight line drawn through X meets the two circles at A and B. If D and C are marked on the circles such that $AD//BC$, show that DYC is a straight line.

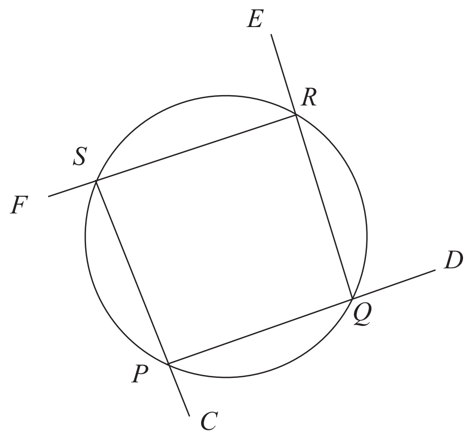


21.3 The relationship between the exterior angle and the interior opposite angle

In the cyclic quadrilateral $ABCD$ shown in the figure, AB is produced to E.



Then, $C\hat{B}E$ is an exterior angle of the cyclic quadrilateral. The corresponding interior opposite angle is $A\hat{D}C$.

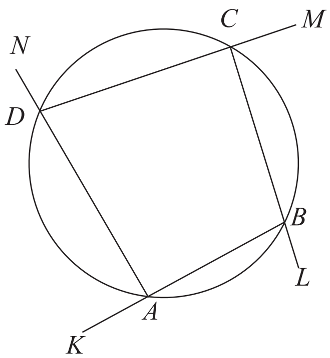


The following table has been filled by considering the cyclic quadrilateral *PQRS* in the figure.

Produced side	**Exterior angle**	**Interior opposite angle**
PQ	$D\hat{Q}R$	$P\hat{S}R$
QR	$E\hat{R}S$	$Q\hat{P}S$
RS	$F\hat{S}P$	$P\hat{Q}R$
SP	$Q\hat{P}C$	$Q\hat{R}S$

The relationship between an exterior angle and the interior opposite angle of a cyclic quadrilateral is given in the theorem below.

Theorem: If one side of a cyclic quadrilateral is produced, the exterior angle so formed is equal to the interior opposite angle of the quadrilateral.



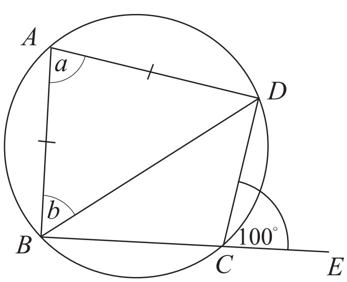
For the above figure, the following are true according to the theorem.

$$D\hat{A}K = B\hat{C}D$$
$$A\hat{B}L = C\hat{D}A$$
$$B\hat{C}M = B\hat{A}D$$
$$C\hat{D}N = A\hat{B}C$$

Let us consider why this theorem is true. Let us for example consider why the angles $D\hat{A}B$ and $B\hat{C}M$ are equal. Since $ABCD$ is a cyclic quadrilateral, $D\hat{A}B + B\hat{C}D = 180^\circ$. Furthermore, since DCM is a straight line, $D\hat{A}B + B\hat{C}D = B\hat{C}D + B\hat{C}M$. Cancelling $B\hat{C}D$ from both sides, we obtain $D\hat{A}B = B\hat{C}M$.

Example 1

Find the values of a and b based on the information in the given figure.

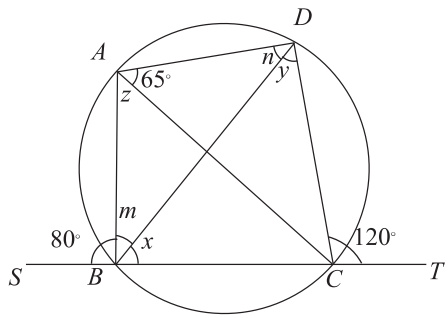


As the exterior angle of the cyclic quadrilateral is equal to the interior opposite angle,
$a = \underline{\underline{100^\circ}}$.

$A\hat{D}B = b$ (since $AB = AD$)
$a + b + b = 180^\circ$ (interior angles of a triangle)
$100^\circ + 2b = 180^\circ$
$b = \underline{\underline{40^\circ}}$

Example 2

Find the values of x, y, z, m and n based on the information in the given figure.



$x = 65^\circ$ (angles in the same segment)

As an exterior angle of a cyclic quadrilateral is equal to the interior opposite angle,

$$B\hat{A}D = D\hat{C}T$$
$$B\hat{A}D = 120^\circ$$
$$z + 65^\circ = 120^\circ$$
$$z = 55^\circ$$
$z = y$ (angles in the same segment)
$$\therefore \; y = \underline{\underline{55^\circ}}$$

As an exterior angle of a cyclic quadrilateral is equal to the interior opposite angle,

$$A\hat{D}C = A\hat{B}S = 80^\circ$$
$$\therefore \; n + y = 80^\circ$$
$$n + 55^\circ = 80^\circ$$
$$n = 80^\circ - 55^\circ$$
$$\therefore \; n = \underline{\underline{25^\circ}}$$

$80^\circ + m + x = 180^\circ$ (angles on a straight line)

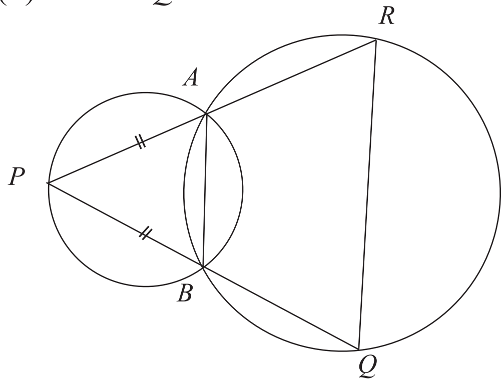
$$\therefore \; 80^\circ + m + 65^\circ = 180^\circ$$
$$m = 180^\circ - 145^\circ$$
$$m = \underline{\underline{35^\circ}}$$

Example 3

The two circles in the given figure intersect at A and B. Moreover, $PA = PB$. If $A\hat{P}B = 70^\circ$,

(i) find the magnitude of $A\hat{R}Q$.

(ii) Is $AB//RQ$?



(i) In the triangle APB,

$P\hat{A}B = P\hat{B}A$ (since $PA = PB$)

$\therefore \; P\hat{A}B = P\hat{B}A = \dfrac{180^\circ - 70^\circ}{2} = 55^\circ$

Moreover, $A\hat{B}P = A\hat{R}Q$ (exterior angle of the cyclic quadrilateral $ABQR$ = the interior opposite angle)

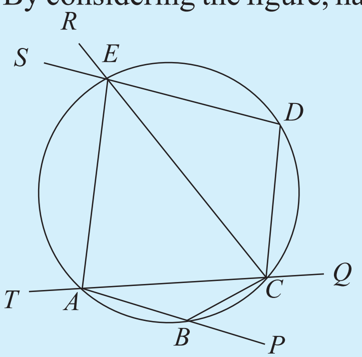
$\therefore \; A\hat{R}Q = \underline{\underline{55^\circ}}$

(ii) $P\hat{A}B = A\hat{R}Q = 55^\circ$.

$\therefore \; AB//RQ$. (since corresponding angles are equal)

Exercise 21.3

1. By considering the figure, name an angle equal to each of the angles given below.

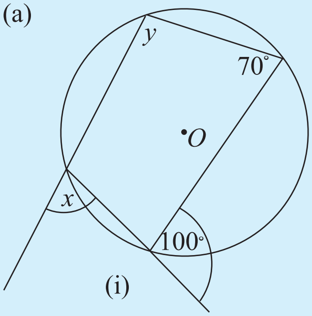


(i) $C\hat{B}P$ (ii) $D\hat{C}Q$ (iii) $R\hat{E}A$

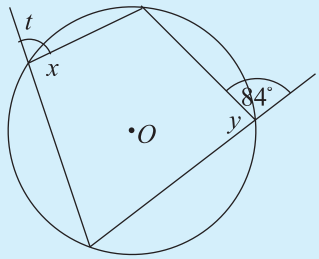
(iv) $S\hat{E}A$ (v) $E\hat{A}T$

2. In each of the circles given below, the centre is O. Find the magnitude of each of the angles denoted by an algebraic symbols.

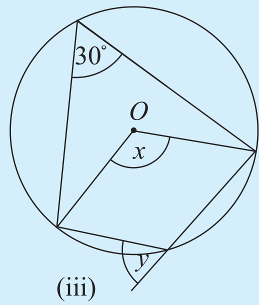
(a)



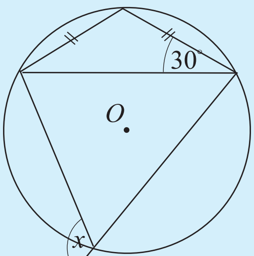
(i)



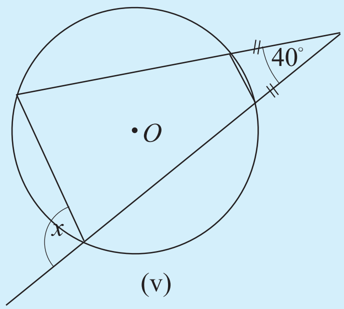
(ii)



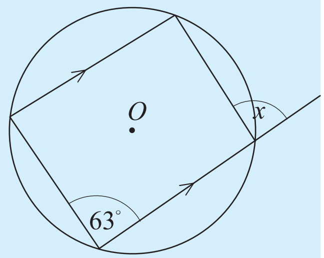
(iii)



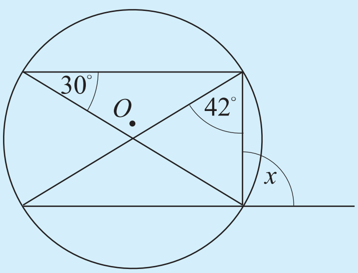
(iv)



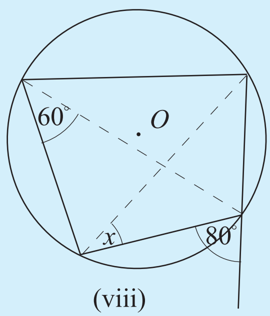
(v)



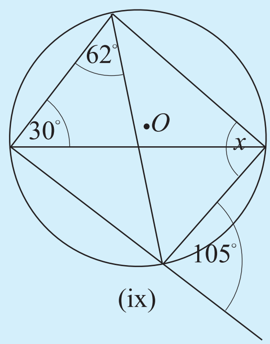
(vi)



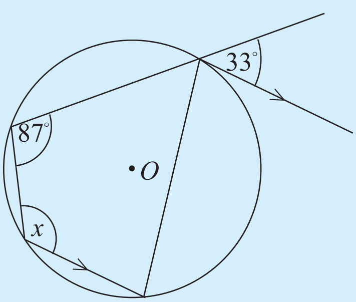
(vii)



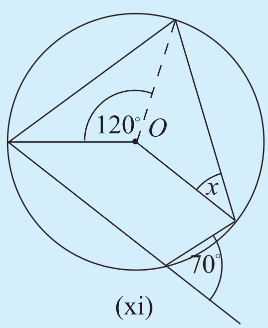
(viii)



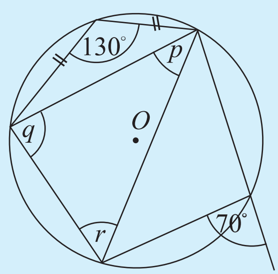
(ix)



(x)

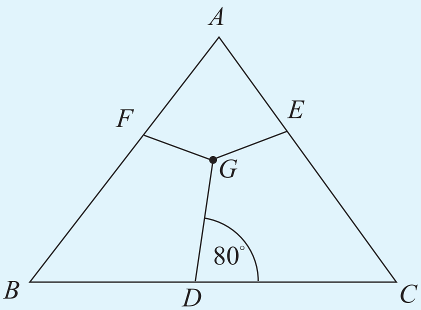


(xi)



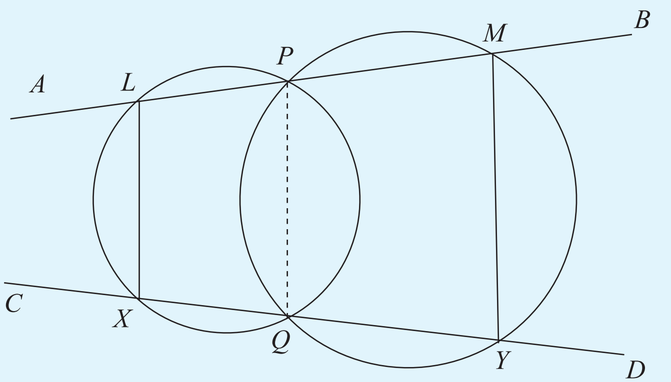
(xii)

3. The points D, E and F are on the sides BC, CA and AB respectively of the triangle ABC such that $BDGF$ and $DCEG$ are cyclic quadrilaterals. Furthermore, $G\hat{D}C = 80^\circ$.



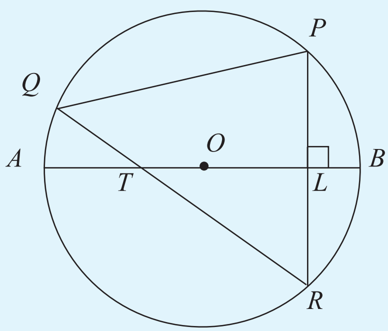
(i) Find the magnitudes of $A\hat{F}G$ and $A\hat{E}G$.
(ii) Show that $AFGE$ is a cyclic quadrilateral.

4. The circles given in the figure intersect at P and Q. The straight lines APB and CQD meet the circles at L, P, M and X, Q, Y respectively.



(i) Find the magnitude of $B\hat{M}Y$ if $A\hat{L}X = 105^\circ$.
(ii) Show that LX and MY are parallel.

5. The centre of the given circle is O. The diameter AB and the chord PR intersect perpendicularly at L. The line segments QR and AB intersects at T.



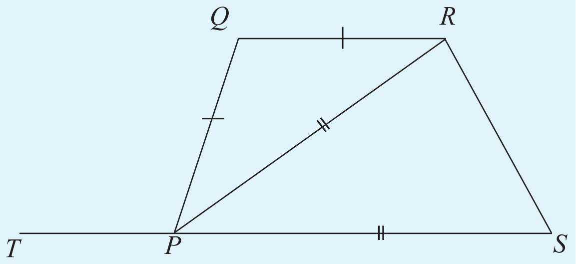
a. If $Q\hat{T}A = x$, write in terms of x,
   (i) the magnitude of $L\hat{R}T$,
   (ii) the magnitude of $O\hat{P}Q$.

b. Show that $QTOP$ is a cyclic quadrilateral.

6. In the figure, $PQ = QR$ and $PR = PS$. If $P\hat{R}S = 2\,Q\hat{R}P$, show that

(i) $PSRQ$ is a cyclic quadrilateral,

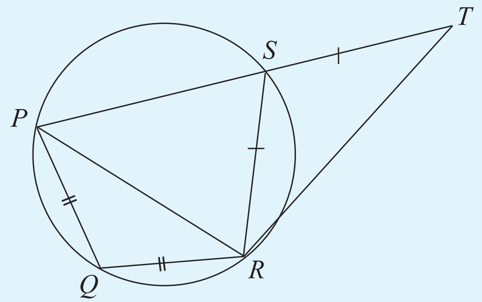
(ii) $Q\hat{P}T : P\hat{R}S = 3 : 2$.



7. $PQ = QR$ in the cyclic quadrilateral $PQRS$. Moreover, PS is produced to T such that $RS = ST$. If $S\hat{R}T = 32^\circ$,

(i) find the magnitude of $Q\hat{R}P$,

(ii) show that QS and RT are parallel.



22

Tangents

By studying this lesson you will be able to,

- identify the tangent which is drawn through a point on a circle and its characteristics,
- identify the tangents drawn to a circle from an external point and their characteristics,
- identify the angles in the alternate segment and solve related problems.

22.1 Tangents

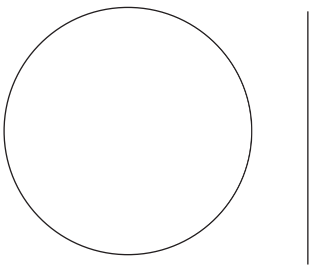


Figure (i)

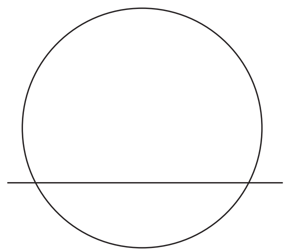


Figure (ii)

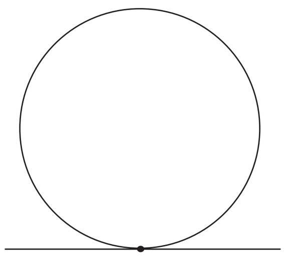


Figure (iii)

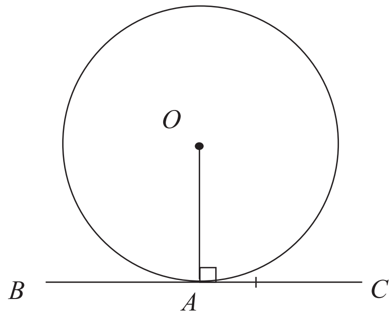
In Figure (i), the circle and the straight line have no points in common. Therefore the straight line is situated external to the circle.

In Figure (ii), the circle and the straight line intersect at two points. Therefore the circle and the straight line have two points in common. The straight line is known as a secant of the circle.

In Figure (iii), the circle and the straight line have one point in common. Here the straight line touches the circle and therefore the straight line is known as a "**tangent**" of the circle. The point which is common to the tangent and the circle is known as the **tangential point**.

Line drawn through a point on a circle, perpendicular to the radius

To learn about the line drawn through a point on a circle which is perpendicular to the radius, consider the facts given below.



O is the centre of the circle in the figure. The radius drawn through the point A which is on the circle is OA. The line drawn perpendicular to the radius OA through A is BC. It is clear that $O\hat{A}C = 90^\circ$ and that BC is a tangent to the circle.
That is,
the line BC drawn perpendicular to the radius OA through the point A is a tangent to the circle.
This result can be written as a theorem as follows.

Theorem: The straight line drawn through a point on a circle and perpendicular to the radius through the point of contact, is a tangent to the circle.

Furthermore the converse of the above theorem is also true.

That is,

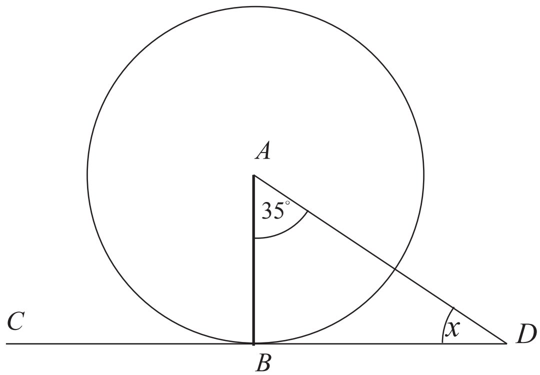
if a tangent is drawn to the circle through a particular point, then the radius that passes through that point and the said tangent are perpendicular to each other.

This result too can be written as a theorem as follows.

Converse of the theorem: The tangent through a point on a circle is perpendicular to the radius drawn to the point of contact.

Example 1

The tangent drawn to the circle with centre A through the point B is CD. If $B\hat{A}D = 35^{o}$, then find the value of x.

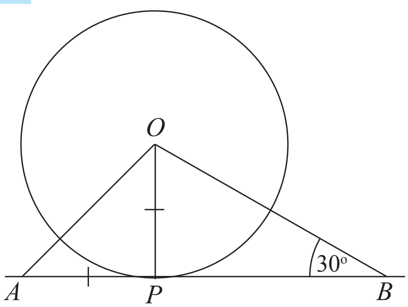


$A\hat{B}D = 90^{o}$ (the tangent through a point on a circle is perpendicular to the radius drawn to the point of contact)

As the sum of the interior angles of a triangle is equal to 180^{o},

$$35^{o} + 90^{o} + x = 180^{o}$$
$$x = 180^{o} - 35^{o} - 90^{o}$$
$$\underline{\underline{x = 55^{o}}}$$

Example 2



In the figure, the tangent drawn to the circle with centre O through P is AB. If $OP = AP$ and $O\hat{B}P = 30^{o}$, find the magnitude of $A\hat{O}B$.

$O\hat{P}A = 90^{o}$ (the tangent through a point on a circle is perpendicular to the radius drawn to the point of contact)

$OP = AP$ (given)

$\therefore P\hat{O}A = P\hat{A}O$ (in an isosceles triangle, the angles opposite equal sides are equal)

In the triangle APO,

$P\hat{A}O + P\hat{O}A + O\hat{P}A = 180^\circ$ (the sum of the interior angles of a triangle is 180°)

$\therefore\ P\hat{A}O + P\hat{O}A + 90^\circ = 180^\circ$

$P\hat{A}O + P\hat{O}A = 180^\circ - 90^\circ$

$P\hat{A}O + P\hat{O}A = 90^\circ$

$\therefore\ 2\,P\hat{A}O = 90^\circ$ (since $P\hat{A}O = P\hat{O}A$)

$$P\hat{A}O = \frac{90^\circ}{2}$$

$$= 45^\circ$$

In the triangle AOB,

$A\hat{O}B + P\hat{A}O + P\hat{B}O = 180^\circ$ (the sum of the interior angles of a triangle is 180°)

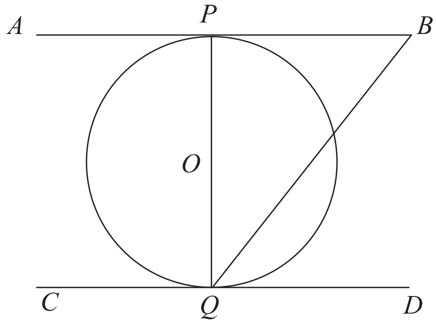
$A\hat{O}B + 45^\circ + 30^\circ = 180^\circ$

$A\hat{O}B + 75^\circ = 180^\circ$

$A\hat{O}B = 180^\circ - 75^\circ$

$= \underline{\underline{105^\circ}}$

Example 3



PQ is a diameter of the circle with centre O. The tangents drawn to the circle through P and Q, are AB and CD respectively. Show that $P\hat{B}Q = B\hat{Q}D$.

Since the tangent drawn to the circle through a point on the circle is perpendicular to the radius drawn to the tangential point,

$Q\hat{P}B = 90^\circ$ and

$P\hat{Q}D = 90^\circ$.

$\therefore\ Q\hat{P}B + P\hat{Q}D = 90^\circ + 90^\circ$

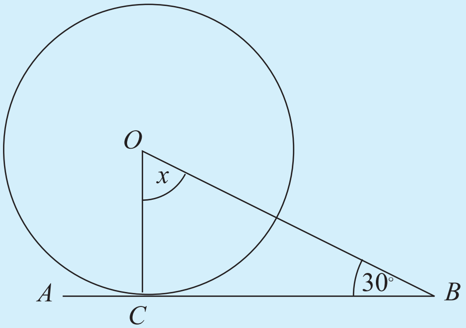
$= 180^\circ$

$\therefore\ AB \,//\, CD$ (allied angles)

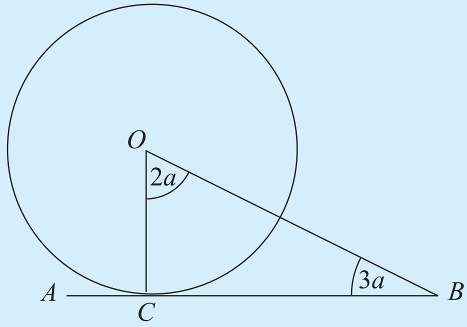
$\therefore\ P\hat{B}Q = B\hat{Q}D$ ($AB \,//\, CD$ and alternate angles)

Exercise 22.1

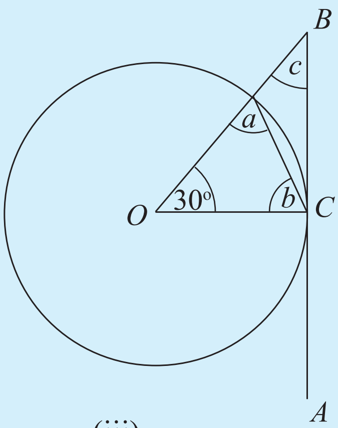
1. In each circle given below, the centre is O and AB is the tangent drawn to the circle through the point C. Find the value of each algebraic symbol based on the data in the figure.



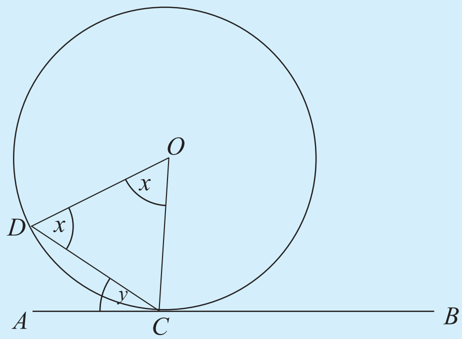
(i)



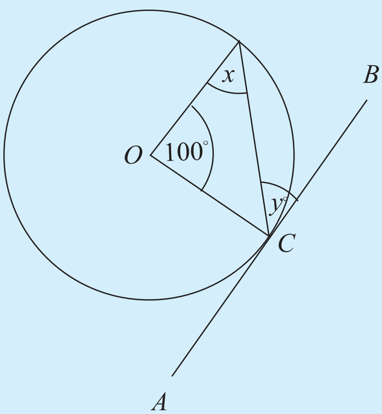
(ii)



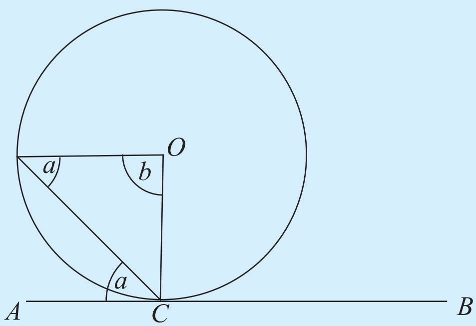
(iii)



(iv)

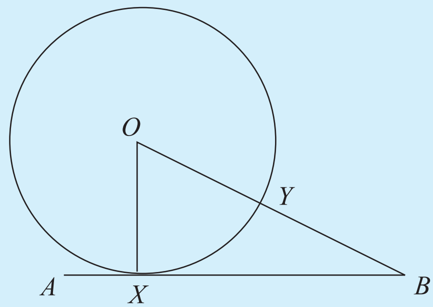


(v)

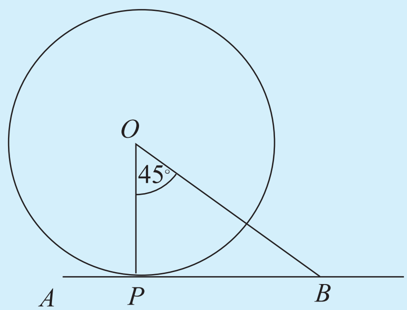


(vi)

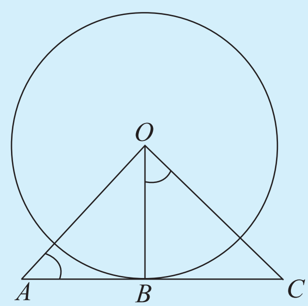
2. In the circle with centre O in the figure, AB is the tangent to the circle through the point X. If the radius of the circle is 6 cm and $YB = 4$ cm, find the length of XB.



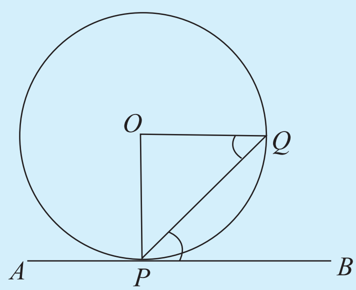
3. The tangent drawn to the circle with centre O through the point P is AB. Find the radius of the circle if $B\hat{O}P = 45^\circ$ and $PB = 6$ cm.



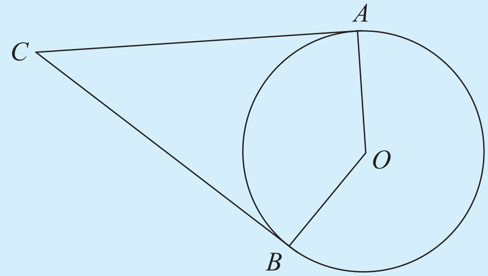
4. The tangent drawn to the circle with centre O through the point B is AC. If $O\hat{A}B = B\hat{O}C$ then show that $A\hat{O}B = B\hat{C}O$.

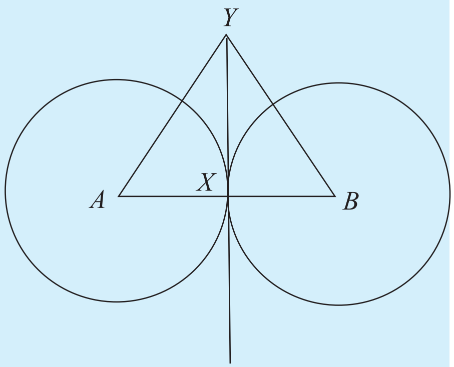


5. The tangent drawn to the circle with centre O though the point P is AB. Q is on the circle such that $O\hat{Q}P = Q\hat{P}B$. Show that OQ and PO are perpendicular to each other.

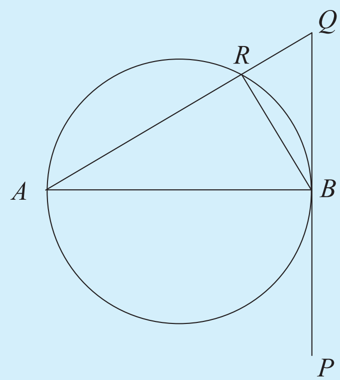


6. The tangents drawn to the circle with centre O through the points A and B intersect at point C. Show that $AOBC$ is a cyclic quadrilateral.





7. Two circles with equal radii and with centres A and B are shown in the figure. Y is situated such that $AY = YB$. Show that YX is a common tangent to the two circles.



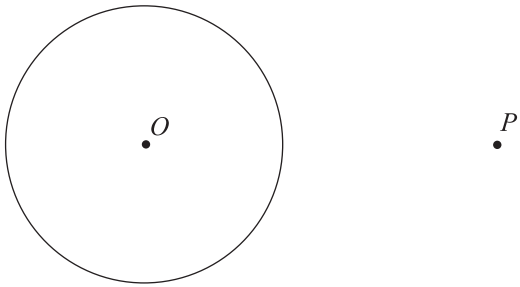
8. In the given figure, AB is a diameter of the circle and PQ touches the circle at the point B. Prove that,

(i) $Q\hat{R}B = 90^\circ$ and

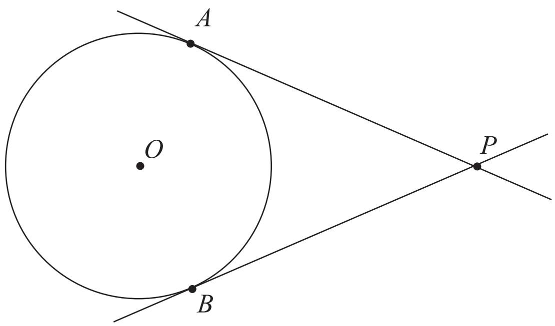
(ii) $A\hat{B}R = R\hat{Q}B$

22.2 Tangents drawn to a circle from an external point

Let us consider a point P which is external to the circle with centre O.

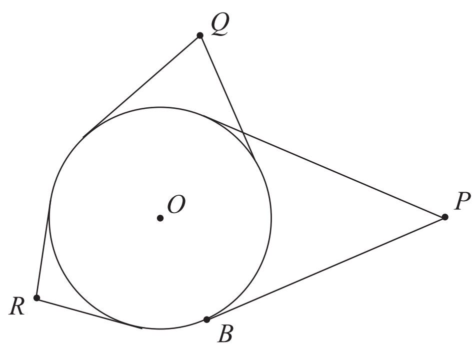


Two tangents to the circle that pass through the point P can be drawn. These two tangents are shown in the following figure.

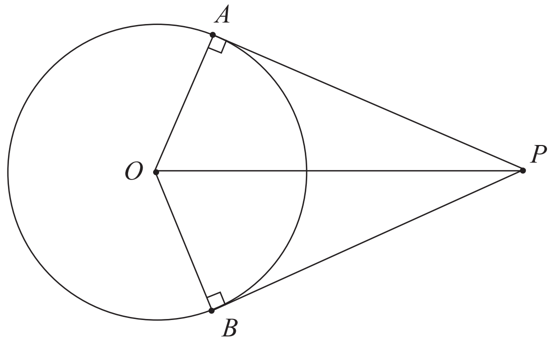


These two tangents are known as the tangents drawn to the circle from the external point P.

Understand that there are two tangents that can be drawn to a circle from an external point P irrespective of where the point is located. In the figure below are three points P, Q and R through which two tangents each are drawn.



Now let us consider the characteristics of the two tangents drawn from an external point to a circle.
Let us mark A and B as the tangential points and OA and OB as the respective radii. Let us also draw the straight line segment OP.



As learnt in section 22.1 above, tangents are perpendicular to the radii drawn through the tangential points. This is indicated in the above figure.

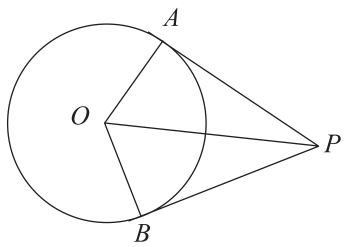
Observing the triangles OAP and OBP, we might guess that they are congruent, based on symmetry. In fact they are congruent. This can easily be proved. Let us see how this is proved. First observe that they are both right angled triangles. Therefore if we show that the hypotenuse and one side of one triangle are equal to the hypotenuse and corresponding side of the other triangle, we can conclude that they are congruent under the RHS (Hyp.S) case. In both triangles, the hypotenuse is the common side OP. Furthermore, since OA and OB are both radii, they are equal in length. Accordingly, these two triangles are congruent under the RHS case. As they are congruent,

(i) $AP = BP$; that is, the tangents are equal in length.
(ii) $A\hat{P}O = B\hat{P}O$; that is, the angle between the two tangents is bisected by OP.
(iii) $A\hat{O}P = B\hat{O}P$; that is, both tangents subtend equal angles at the centre of the circle.

The above considered facts are expressed as a theorem as follows.

Theorem: If two tangents are drawn to a circle from an external point, then,
(i) the two tangents are equal in length.
(ii) the angle between the tangents is bisected by the straight line joining the external point to the centre.
(iii) the tangents subtend equal angles at the centre.

Let us now consider how this theorem is proved formally.



Data : The tangents drawn from the external point P through the points A and B on the circle with centre O are AP and BP respectively.

To prove :

(i) $AP = BP$

(ii) $A\hat{P}O = B\hat{P}O$

(iii) $P\hat{O}A = P\hat{O}B$

Proof : $O\hat{A}P = O\hat{B}P = 90^{\circ}$ (tangents are perpendicular to the radii)

$\therefore$ POA and POB are right angled triangles.

Now, in the triangles POA and POB,

$OA = OB$ (radii of the same circle)

OP is the common side

$\therefore$ $\Delta POA \equiv \Delta POB$ (Hyp.S)

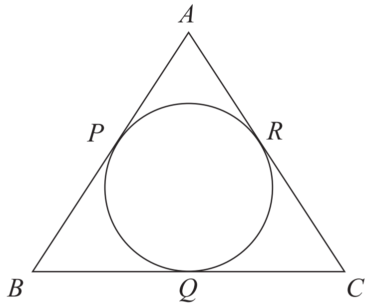
In congruent triangles, corresponding sides and corresponding angles are equal.

$\therefore$ (i) $AP = BP$

$\therefore$ (ii) $A\hat{P}O = B\hat{P}O$

$\therefore$ (iii) $P\hat{O}A = P\hat{O}B$

Example 1



As shown in the figure, the circle touches the triangle ABC at the points P, Q and R. If $AB = 11$cm and $CR = 4$ cm, find the perimeter of the triangle ABC.

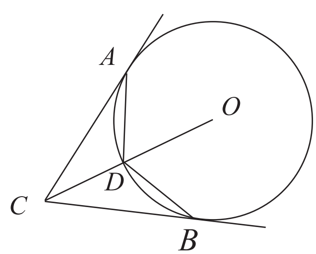
Tangents drawn from an external point to a circle are equal in length.

$$\therefore \; AP = AR$$
$$BP = BQ$$
$$CR = CQ$$

$$\begin{aligned}
\therefore \text{ The perimeter of } \Delta\; ABC &= AB + BC + CA \\
&= 11 + (BQ + QC) + (CR + RA) \\
&= 11 + (BP + CR) + (CR + AP) \\
&= 11 + (BP + 4) + (4 + AP) \\
&= 19 + (BP + AP) \\
&= 19 + AB \\
&= 19 + 11 \\
&= 30
\end{aligned}$$

$\therefore$ The perimeter of the triangle ABC is 30 cm.

Example 2



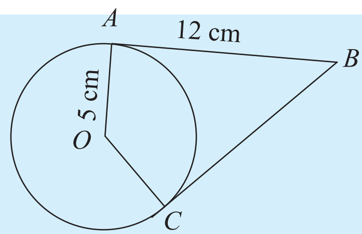
As shown in the figure, the two tangents drawn from the external point C to the circle with centre O touches the circle at A and B. The line drawn from the centre O to C intersects the circle at D.
Show that $AD = BD$.

We can obtain the required result by proving that the two triangles ACD and BCD are congruent.

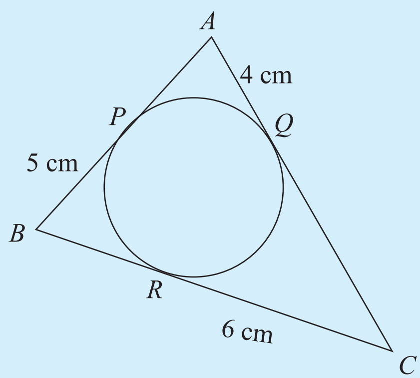
In the triangles ACD and BCD,
$AC = BC$ (tangents drawn to a circle from an external point are equal in length)
$A\hat{C}O = B\hat{C}O$ (the angle between the tangents is bisected by the straight line joining the external point to the centre)
CD is the common side
$\therefore \ \Delta ACD \equiv \Delta BCD$ (SAS)
$\therefore \ \underline{\underline{AD = BD}}$ (corresponding sides of two congruent triangles)

Exercise 22.2

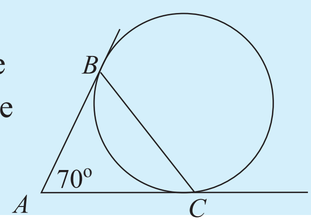
1. The tangents through the points A and C on the circle with centre O in the figure, meet at B. If the radius of the circle is 5 cm and AB = 12 cm, then find the perimeter of the quadrilateral $ABCO$.



2. The tangents drawn to the circle through the points P, Q and R on the circle are AB, AC and BC respectively. Find the perimeter of the triangle ABC if RC = 6 cm, BP = 5 cm and AQ = 4 cm.

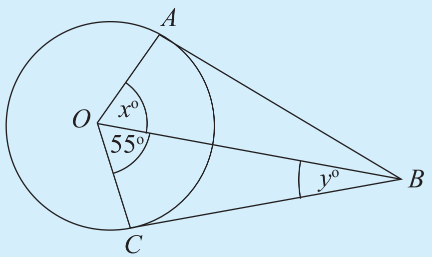


3. The tangents drawn to the circle in the figure through the points B and C meet at A. If $B\hat{A}C = 70^\circ$, find the magnitude of $A\hat{B}C$.

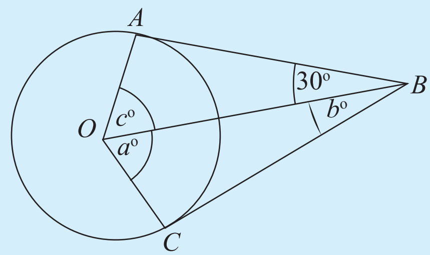


4. The centre of each of the circles shown below is O. The tangents drawn to the circle through the points A and C meet at B. Based on the given data, find the values represented by the algebraic symbols.

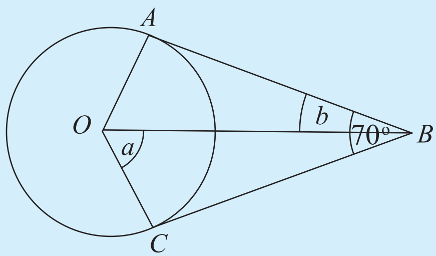
(i)



(ii)

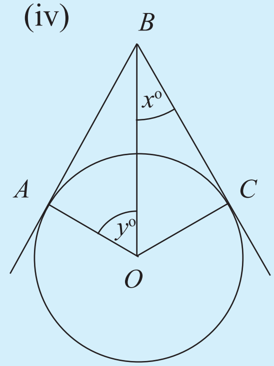


(iii)



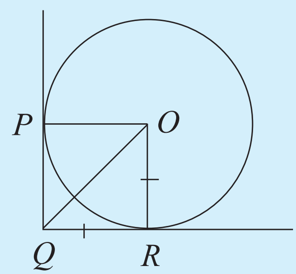
$A\hat{B}C = 70^\circ$

(iv)

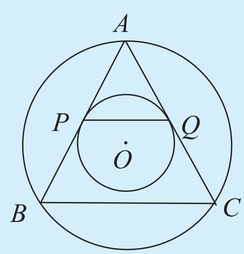


$A\hat{O}C = 110^\circ$

5. O is the centre of the circle shown in the figure. The tangents drawn to the circle through the points P and R meet at Q. If $QR = OR$, show that $PQRO$ is a square.



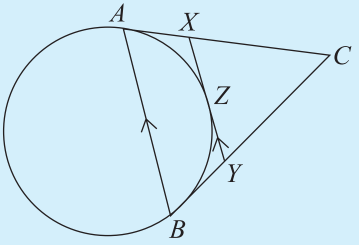
6. A, B and C are points on the larger circle with centre O shown in the figure. AB and AC touch the smaller circle at the points P and Q. Show that,



(i) APQ is an isosceles triangle,

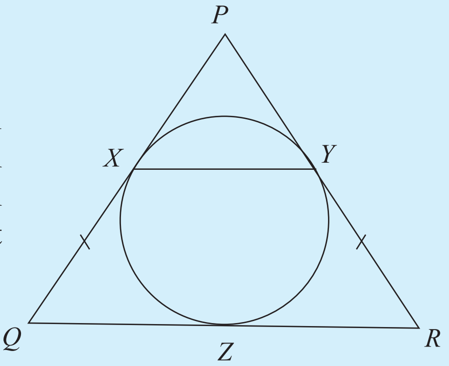
(ii) $BC \,//\, PQ$.

7. According to the information in the figure, show that $XC = CY$. Also, AC and BC are the tangents drawn to the given circle at A and B respectively.



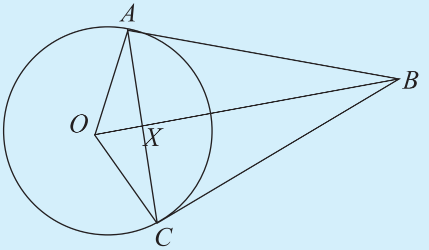
8. The tangents drawn from the external point P to the circle in the figure, touch the circle at X and Y. The line QR which touches the circle at Z is drawn such that $XQ = YR$. Show that,

(i) $PR = PQ$,
(ii) $QR = XQ + YR$,
(iii) $XY \parallel QR$.



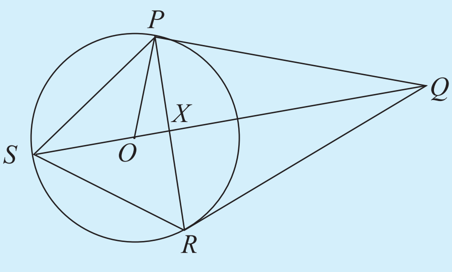
9. The tangents drawn through the points A and C on the circle with centre O shown in the figure meet at the point B. Show that,

(i) $OAX\,\Delta \equiv OCX\,\Delta$,
(ii) OB is the perpendicular bisector of AC,
(iii) $A\hat{O}C = 2\,A\hat{C}B$.



10. The tangents drawn from the external point Q to the circle with centre O shown in the figure are PQ and QR. QO produced meets the circle at S. Show that,

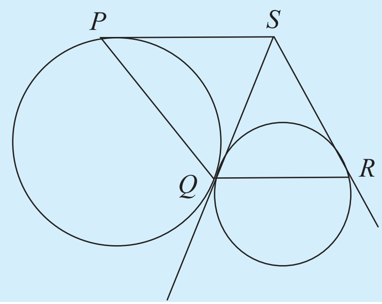
(i) $\Delta\, PQS \equiv \Delta\, QRS$
(ii) $2\,O\hat{P}X = P\hat{Q}R$.



11. As shown in the figure, Q is a point on both circles and QS is a tangent to both circles. The other tangents drawn from S to the two circles touch the circles at P and R. Show that,

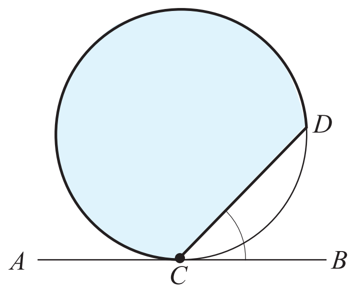
(i) $PS = SR$,

(ii) $P\hat{Q}R = S\hat{P}Q + S\hat{R}Q$.



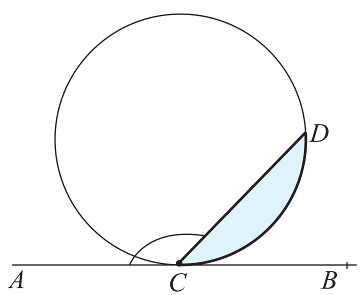
22.3 Angles in the alternate segment

First let us see what an alternate segment is. To do this let us consider the following figure.



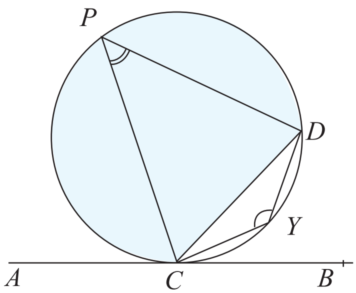
As shown in the figure, the straight line AB touches the circle at the point C. In the figure, CD is a chord. The circle is divided into two segments by the chord CD. One segment is the portion which is shaded light blue. The other segment is the smaller portion which is not shaded. There are two angles formed by the chord CD meeting the tangent AB. One angle is $A\hat{C}D$. The other is $B\hat{C}D$ The alternate segment corresponding to the angle $B\hat{C}D$ is the portion shaded light blue. Similarly, the alternate segment corresponding to the angle $A\hat{C}D$ is the portion of the circle which is not shaded.

In the figure given below, the alternate segment corresponding to $A\hat{C}D$ is shaded light blue.



Theorems related to angles in the alternate segment

Consider the figure given below. $C\hat{P}D$ is in the larger segment which is shaded light blue. Therefore the angle $C\hat{P}D$ is in the alternate segment of $D\hat{C}B$. Similarly $C\hat{Y}D$ is in the alternate segment of $A\hat{C}D$.

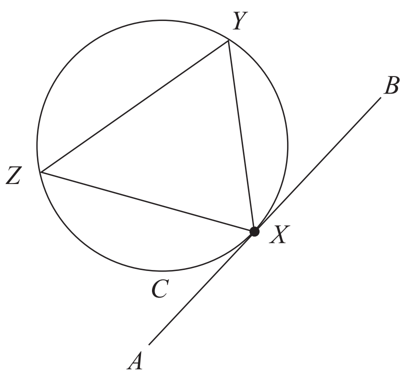


There is a special relationship involving tangents of a circle. For the above figure, it means that the angles $D\hat{C}B$ and $C\hat{P}D$ are equal, and also that the angles $A\hat{C}D$ and $C\hat{Y}D$ are equal. In other words, "The angles which a tangent to a circle makes with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle". This is a very important result. Therefore let us write it as a theorem and remember it.

Theorem: The angles which a tangent to a circle makes with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle**.**

To check the validity of the above theorem let us do some activities.

Activity 1

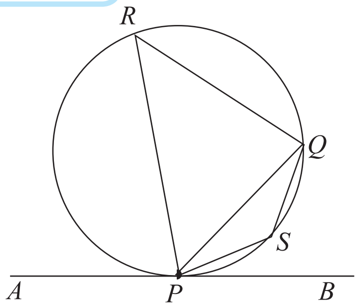


- Draw a circle and mark a point X on it.
- Draw a line touching the circle at X (draw a radius from X and draw a line

perpendicular to it.) Name it AB.

- Mark two other points on the circle and name them Y and Z.
- As shown in the figure, join the points X, Y and Z.
- Using the protractor, measure the magnitude of $B\hat{X}Y$ and also the magnitude of $X\hat{Z}Y$, which is the corresponding angle in the alternate segment, and check whether they are equal.
- Similarly, find the magnitude of $A\hat{X}Z$ and the magnitude of $X\hat{Y}Z$, which is the corresponding angle in the alternate segment, and check whether they are equal

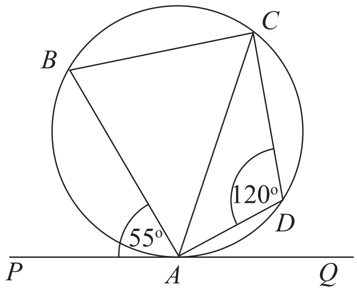
Activity 2



- Draw a circle and mark a point on it and name it P. Draw a line touching the circle at P (this can be done by drawing a radius from P and drawing a line perpendicular to it through P). Name it AB.
- Draw a chord from P and name it PQ.
- Mark two points on the circle on opposite sides of the chord and name them R and S.
- Draw the line segments QR, QS, PS and PR.
- Using the protractor, measure the magnitude of $B\hat{P}Q$ and the magnitude of $P\hat{R}Q$, which is the angle in the alternate segment, and check whether they are equal.
- Similarly measure the magnitude of $A\hat{P}Q$ and the magnitude of $P\hat{S}Q$, which is the angle in the alternate segment, and check whether they are equal.

From the above activities you would have understood that the angles which a tangent to a circle makes with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.

Example 1



In the above figure, the line PQ touches the circle at A. Furthermore, B, C and D are also on the circle. $P\hat{A}B = 55^\circ$ and $A\hat{D}C = 120^\circ$. Find the magnitude of $B\hat{A}C$.

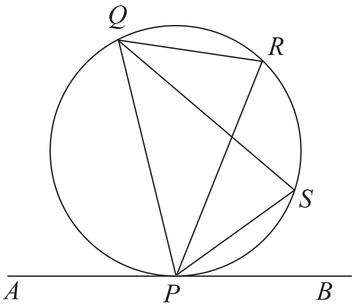
Initially let us find the magnitude of the angle $P\hat{A}C$.

$P\hat{A}C = A\hat{D}C$ (the angles which a tangent to a circle makes with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle)

$$P\hat{A}B + B\hat{A}C = 120^\circ$$
$$55^\circ + B\hat{A}C = 120^\circ$$
$$B\hat{A}C = 120^\circ - 55^\circ$$
$$= \underline{\underline{65^\circ}}$$

Example 2

The straight line AB touches the circle at point P, Q and R are points on the circle. The bisector of the angle $P\hat{Q}R$ meets the circle at S. Show that PS is the bisector of the angle $B\hat{P}R$.



$B\hat{P}S = P\hat{Q}S$ (the angles which a tangent to a circle makes with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle)

$R\hat{P}S = R\hat{Q}S$ (angles in the same segment are equal)

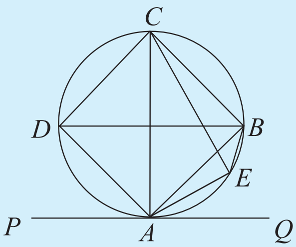
$P\hat{Q}S = R\hat{Q}S$ (data, because QS bisects the angle $P\hat{Q}R$)

$\therefore\ B\hat{P}S = R\hat{P}S$

$\therefore\ PS$, is the angle bisector of $B\hat{P}R$

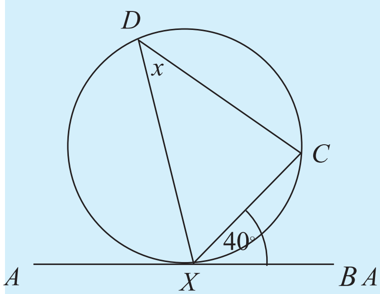
Exercise 22.3

1. PQ is a tangent to the circle through the point A. The points B, C, D and E lie on the circle.

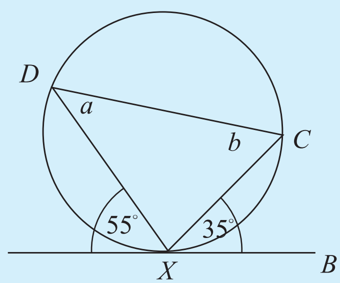


Angle between the tangent and the chord	Angle in the alternate segment
$B\hat{A}Q$	....
$P\hat{A}B$	....
$P\hat{A}D$	....
$E\hat{A}Q$	....
....	$D\hat{B}A$
....	$D\hat{C}A$

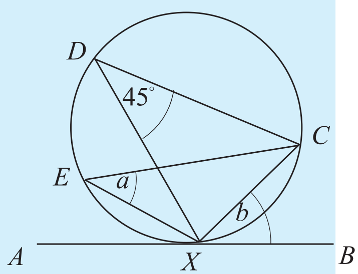
2. In each of the following figures, AB is the tangent to the circle drawn through the point X. Find the values represented by the algebraic symbols.



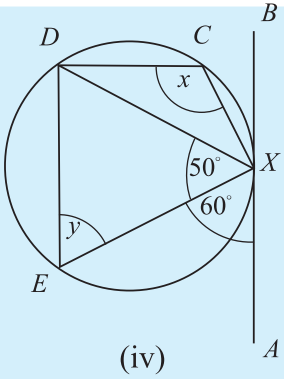
(i)

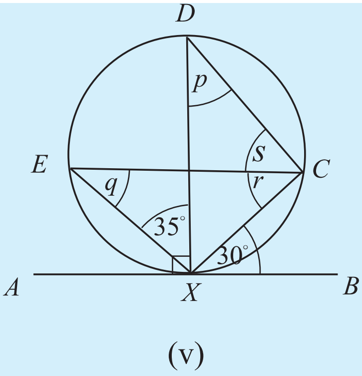


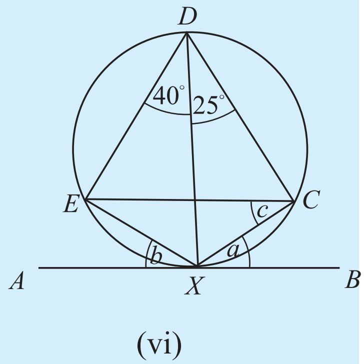
(ii)



(iii)

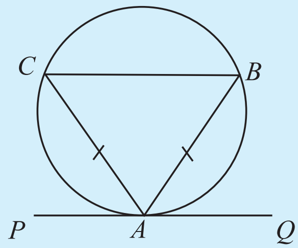






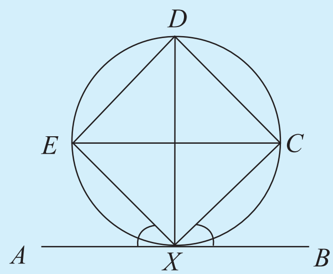
3. *PQ* is the tangent to the circle through *A*. If $AC = AB$ show that,

(i) $C\hat{A}P = B\hat{A}Q$,
(ii) $PQ \parallel CB$.



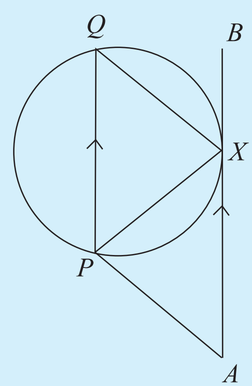
4. *AB* is a tangent drawn to the circle through *X*. The points *C* and *E* are on the circle such that $B\hat{X}C = A\hat{X}E$. *D* is another point on the circle. Show that,

(i) *XD* is the bisector of $E\hat{D}C$,
(ii) $EX = CX$,
(iii) $AB \parallel EC$.



5. The straight line *AB* touches the circle *at X*. The chord *PQ* is drawn such that $PQ \parallel AB$. Prove that,

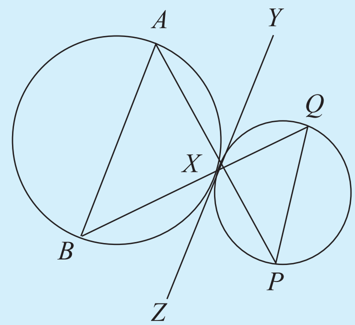
(i) $B\hat{X}Q = A\hat{X}P$,
(ii) *AXQP* is a parallelogram if $PX = PA$.



6. Two circles touch each other externally at X. YZ is the common tangent. AB is a chord of one circle. AX produced and BX produced meet the other circle at P and Q respectively. Show that,

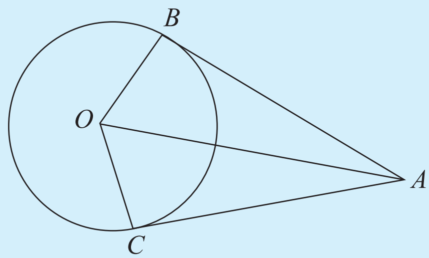
(i) $B\hat{X}Z = X\hat{P}Q$,

(ii) $AB \parallel PQ$.

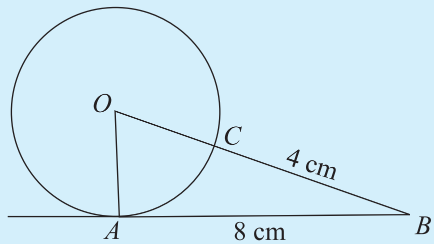


Miscellaneous Exercise

1. The tangents drawn to a circle with centre O from an external point A meet the circle at B and C. If the radius of the circle is 5 cm and OA is 13 cm, find the area of the quadrilateral $OBAC$.



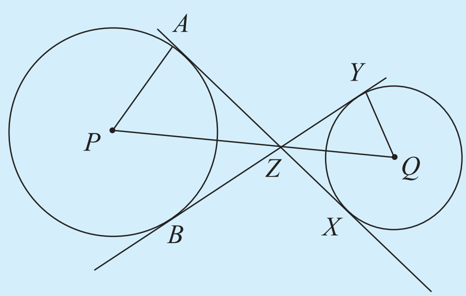
2. AB is the tangent drawn to the circle with centre O through the point A. OB intersects the circle at C. If $CB = 4$ cm and $AB = 8$ cm, find the radius of the circle.



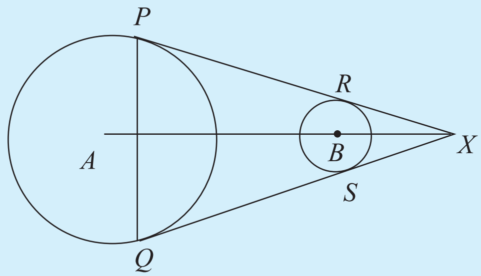
3. The centres of the two circles shown in the figure are P and Q. The two tangents drawn to the larger circle through A and B touch the smaller circle at X and Y respectively. Moreover, these two tangents intersect at Z. Show that,

(i) $AX = BY$,

(ii) $A\hat{P}Z = Y\hat{Q}Z$.



4. As shown in the figure, the tangents PX and QX touch the circles at P, R, Q and S. The centres of the circles are A and B. Show that,



(i) $PR = QS$,
(ii) $PQ \parallel RS$,
(iii) A, B and X are on the same straight line.

23 Constructions

By studying this lesson you will be able to,

- do constructions related to straight lines and angles,
- construct circles related to triangles,
- construct tangents to circles.

23.1 Constructions related to straight lines and angles

Let us learn some constructions which will be required in the constructions that will be studied later on. We use only a pair of compasses and a straight edge to do constructions.

1. Construction of the perpendicular bisector of a straight line segment

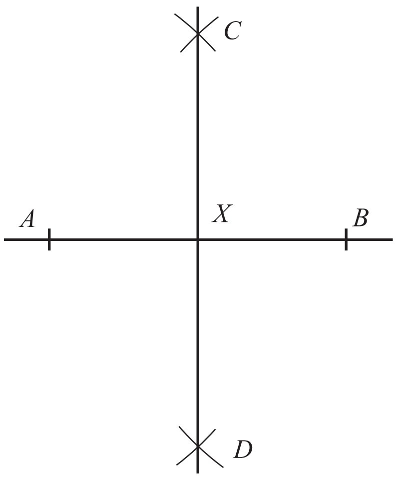
The perpendicular bisector of a straight line segment is the line drawn perpendicular to the straight line segment through its midpoint.
Let us consider a straight line segment AB.

Step 1: Take a length of a little more than half the length of AB onto the pair of compasses. Taking A as the centre, draw two arcs above and below the straight line segment.

Step 2: Using the same length (i.e., without altering the pair of compasses) and taking B as the centre, draw another two arcs such that they intersect the two arcs drawn earlier.

Step 3: Name the two points of intersection as C and D. Draw a straight line segment joining C to D.

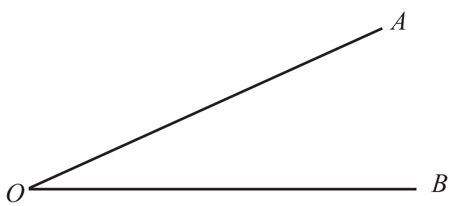
Step 4: Name the point where this straight line segment intersects AB as X.



CD is the perpendicular bisector of the straight line segment AB. Using a protractor, measure the magnitudes of the angles $A\hat{X}C$, $B\hat{X}C$, $A\hat{X}D$ and $B\hat{X}D$, and using a cm/mm scale, measure the lengths of AX and BX. Thereby establish the fact that CD is the perpendicular bisector of AB.

2. Construction of an angle bisector

Consider the angle $A\hat{O}B$

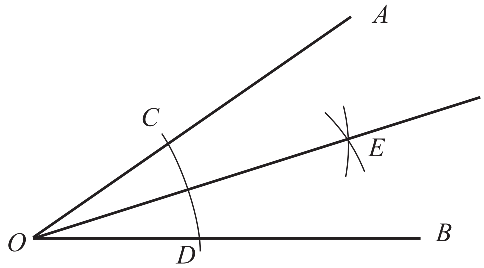


Step 1: Take a length which is less than OA and OB onto the pair of compasses. Taking O as the centre, draw an arc such that it intersects both OA and OB.

Step 2: Name the two points of intersection of the arc with OA and OB as C and D.

Step 3: Taking a suitable length onto the pair of compasses, and taking C and D as the centres, draw two arcs which intersect each other. Name the point of intersection of the two arcs as E.

Step 4: Join O and E.



OE is the angle bisector of $A\hat{O}B$. Establish this fact by measuring the magnitudes of the angles $A\hat{O}E$ and $B\hat{O}E$.

3. Construction of a perpendicular to a line through a given point on the line.

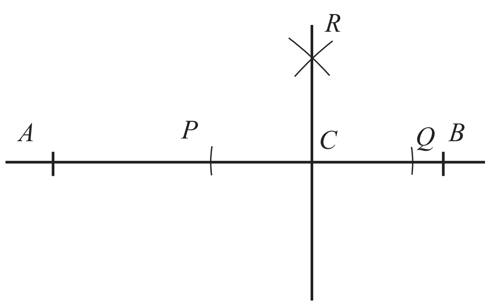
Let us assume that we want to draw a perpendicular to AB through the point C which is on AB.

Step 1: Take a suitable length onto the pair of compasses and taking C as the centre, draw two arcs that intersect the straight line segment AB on the two sides of C.

Step 2: Name the two points of intersection as P and Q.

Step 3: Taking P and Q as the centres and using a fixed radius, draw two arcs above or below the line AB such that they intersect each other.

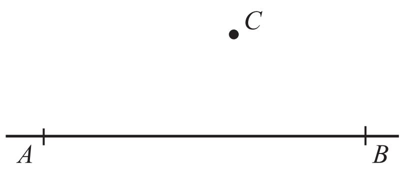
Step 4: Name the point of intersection of the two arcs as R, and join CR with a straight line.



CR is the perpendicular drawn to AB through C. Measure the angles $A\hat{C}R$ and $B\hat{C}R$ and establish this fact.

4. Construction of a perpendicular to a straight line segment from an external point

Let AB be a straight line segment and C an external point.

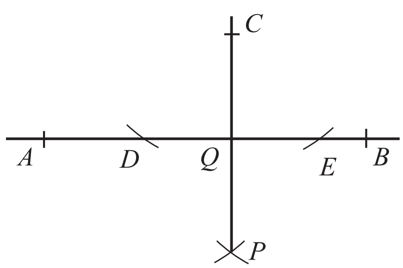


Step 1: Take a length which is a little more than the distance from C to AB onto the pair of compasses. Taking C as the centre, draw two arcs which intersect AB.

Step 2: Name the two points of intersection as D and E.

Step 3: Taking the same radius (or another suitable one), draw two intersecting arcs taking D and E as the centres, on the side of AB opposite to that on which C lies.

Step 4: Name the point of intersection of the two arcs as P and join CP. Name the point of intersection of CP and AB as Q.



CP is the perpendicular drawn to AB from the point C. This can be established by measuring the magnitudes of the angles $C\hat{Q}A$ and $C\hat{Q}B$ using the protractor.

Exercise 23.1

1. Construct the perpendicular bisector of the straight line segment AB where $AB = 5.2$ cm.

2. Construct an angle of 90° and construct its bisector.

3. Construct the triangle ABC where $AB = 6$ cm, $A\hat{B}C = 60^{\circ}$ and $BC = 5$ cm. Construct the perpendicular bisector of AB.

4. (i) Construct the triangle PQR where $PQ = 7$cm, $QR = 6.5$ cm and $PR = 5$ cm.
(ii) Construct the bisectors of $Q\hat{P}R$ and $P\hat{Q}R$.

5. (i) Draw the straight line segment XY of length 5.5 cm.
(ii) Construct a perpendicular to XY through X.
(iii) Mark a point 4 cm from X on the perpendicular and name it Z. Join YZ. Construct a perpendicular from X to YZ.

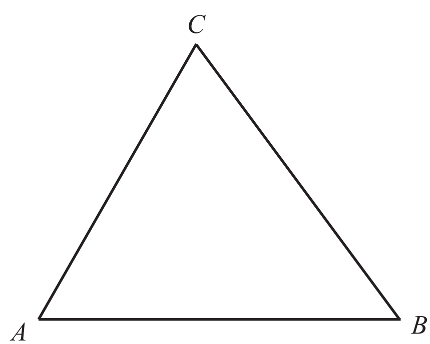
6. (i) Construct an equilateral triangle ABC of side length 6 cm.
(ii) Construct a perpendicular from each vertex of traingle ABC to the opposite side.

23.2 Construction of circles related to triangles

You have learnt earlier how to construct triangles using a straight edge and a pair of compasses when lengths of the sides of the triangle and magnitudes of the angles are given. Now let us learn how to construct circles related to triangles in three cases, using only a straight edge and a pair of compasses.

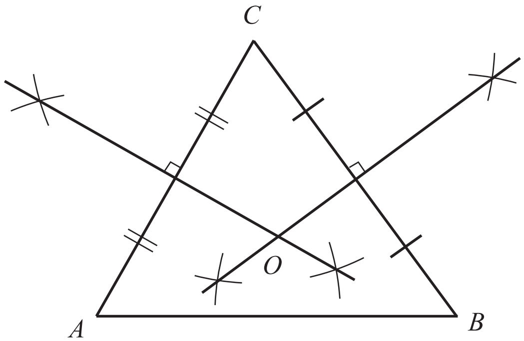
Construction of the circumcircle (circumscribed circle) of a triangle

Draw a triangle and name it ABC.

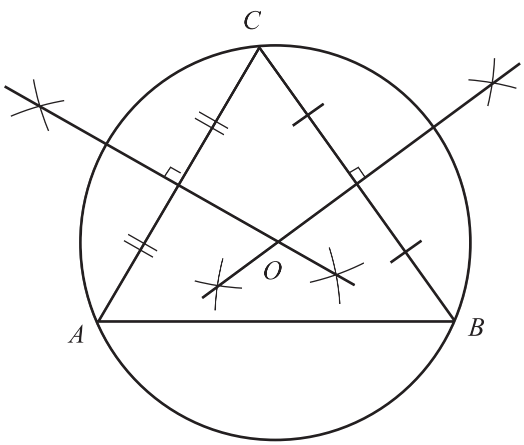


Step 1: Using the pair of compasses, draw the perpendicular bisectors of any two of the three sides *AB, BC* and *CA* of the triangle *ABC*.

Step 2: Name the point of intersection of these perpendicular bisectors as *O*.



Step 3: Taking *O* as the centre and the distance from any vertex of the triangle to *O* as the radius, draw a circle.



Observe that the constructed circle passes through all the vertices of the triangle *ABC*. This circle is known as the circumcircle of the triangle *ABC*. The centre of this circle is known as the circumcentre of the triangle.

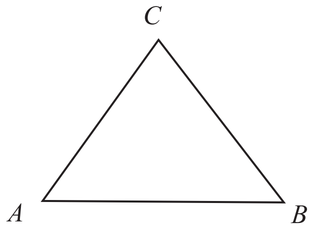
Draw a right angled triangle and an obtuse angled triangle and construct the circumcircle of each triangle.

Using the above constructions, fill in the table given below.

Triangle	Location of the circumcentre		
	Inside the triangle	On a side of the triangle	Outside the triangle
Acute angled triangle	✓	✗	✗
Right angled triangle			
Obtuse angled triangle			

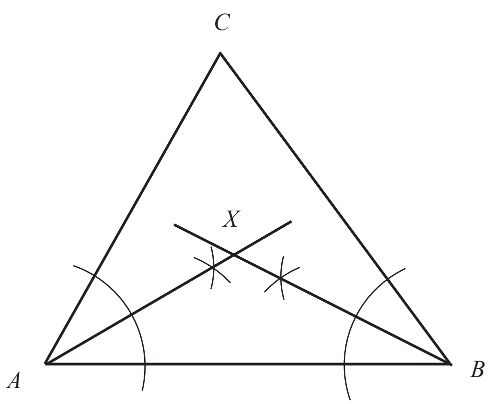
Construction of the incircle (inscribed circle) of a triangle

Draw a triangle and name it *ABC*.

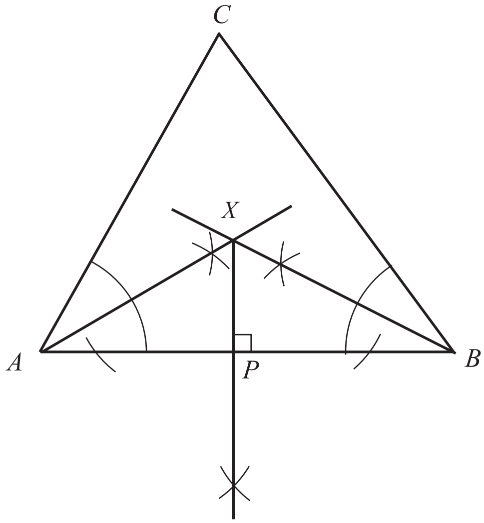


Step 1: Using the pair of compasses construct the angle bisectors of any two interior angles of the triangle *ABC*.

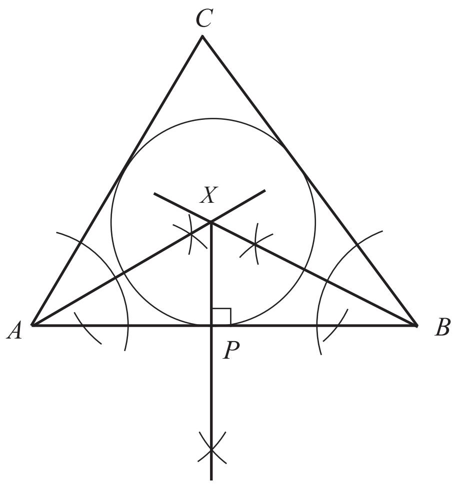
Step 2: Name the point at which the angle bisectors meet as *X*.



Step 3: Construct a perpendicular from *X* to any one of the three sides of the triangle. Name the foot of that perpendicular as *P*.



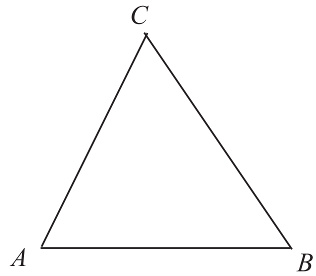
Step 4: Taking XP as the radius and X as the centre, draw a circle.



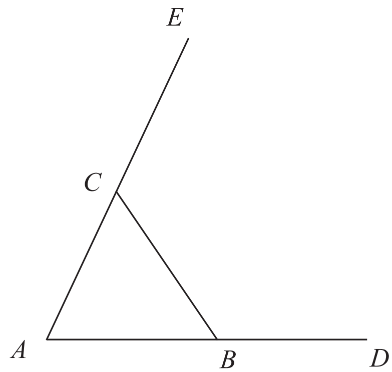
You can see that the circle which was constructed touches the three sides *AB, BC* and *CA* internally. Therefore it is called the **incircle** of the triangle *ABC*. The centre of this circle is known as the **incentre** of the triangle.

Construction of the excircle (escribed circle) of a triangle

Let us consider the triangle ABC.

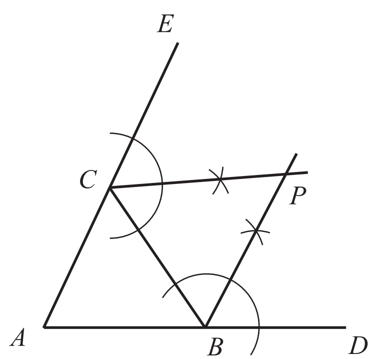


Step 1: Produce the side AB to D and the side AC to E.

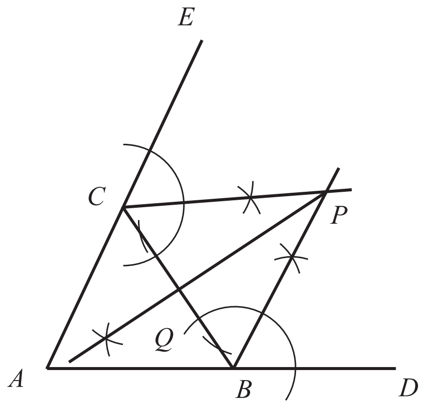


Step 2: Using the pair of compasses, construct the angle bisectors of the angles $C\hat{B}D$ and $B\hat{C}E$.

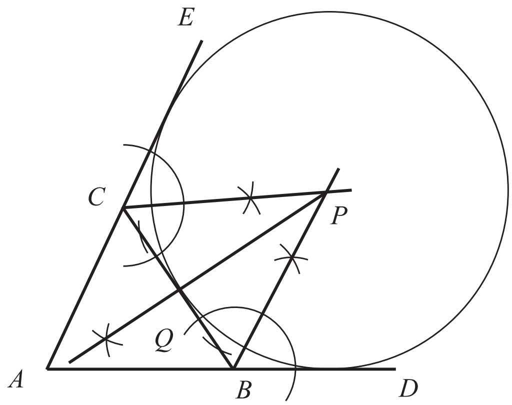
Step 3: Name the point of intersection of the two bisectors as P.



Step 4: Construct a perpendicular from P to BC (or to CE or BD). Name the foot of the perpendicular as Q.



Step 5: Taking P as the centre, draw the circle with radius PQ.



Observe that the circle touches the side BC and the sides AC and AB produced externally. Therefore it is named the **excircle** of the triangle ABC. Its centre is known as the **excentre** of the triangle.

Note: The excircle that touches the side AB and the sides CB and CA produced, as well as the excircle that touches the side CA and the sides BA and BC produced can also be constructed similarly. It is clear therefore that each triangle has three excircles.

Exercise 23.2

1. (i) Construct the triangle ABC where $AB = 5$ cm, $BC = 4.5$ cm and $AC = 4$ cm.
   (ii) Construct the perpendicular bisectors of the sides BC and AC. Name the point where they meet as O.
   (iii) Construct the circumcircle of the triangle ABC.

2. (i) Construct the triangle PQR where $PQ = 6$ cm, $P\hat{Q}R = 90^{\circ}$ and $QR = 4$ cm.
   (ii) Construct the circumcircle of PQR.

3. (i) Construct the triangle XYZ where $XY = 4.2$cm $Y\hat{X}Z = 120^{\circ}$ and $X\hat{Y}Z = 30^{\circ}$.
   (ii) Construct the circumcircle of XYZ.
   (iii) Measure the radius of the circumcircle and write it down.

4. (i) Construct the triangle ABC where $AB = 7$ cm, $BC = 6$ cm and $AC = 5.5$ cm.
   (ii) Construct the angle bisectors of the angles $A\hat{B}C$ and $B\hat{A}C$.
   (iii) Name the point of intersection of the two angle bisectors as P.
   (iv) Draw the incircle of the triangle ABC.

5. (i) Construct the triangle KLM where $KL = 6$ cm, $L\hat{K}M = 105^{\circ}$ and $KM = 9$ cm.
   (ii) Construct the incircle of the triangle KLM and measure and write down its radius.

6. (i) Construct the triangle CDE where $CD = 5.5$ cm, $C\hat{D}E = 60^{\circ}$ and $DE = 4$ cm.
   (ii) Produce CD to P where $DP = 2.8$ cm and CE to Q where $EQ = 2.5$ cm.
   (iii) Construct the bisectors of $E\hat{D}P$ and $D\hat{E}Q$. Name the point where they intersect as X.
   (iv) Construct a perpendicular from X to DE and name the point where it meets DE as K.
   (v) Taking X as the centre, draw a circle with radius XK.

7. (i) Construct the parallelogram $ABCD$ where $AB = 6.2$ cm, $A\hat{B}C = 120^{\circ}$ and $BC = 4.5$ cm.
   (ii) Produce the sides AB and AC and draw an excircle of the triangle ABC.
   (iii) Measure and write down the radius of this circle.

23.3 Construction of a tangent to a circle

Let us recall two theorems we learnt in the lesson on tangents.

1. The straight line drawn through a point on a circle and perpendicular to the radius through the point of contact is a tangent to the circle.

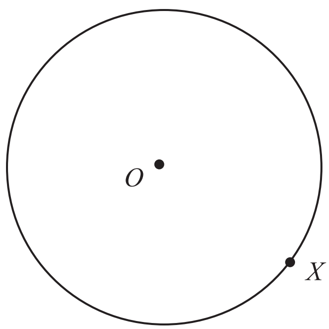
2. The tangents drawn to a circle from an external point (exterior) are equal in length.

Now let us consider how to construct tangents to circles using the above theorems.

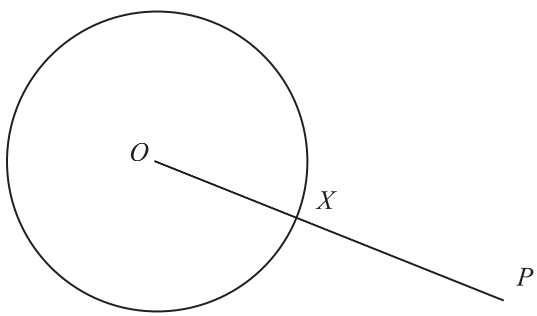
Construction of a tangent to a circle through a point on the circle

To construct this, let us use the theorem, "The straight line drawn through a point on a circle and perpendicular to the radius through the point of contact is a tangent to the circle."

Let the centre of the given circle be O and X be a point on it.

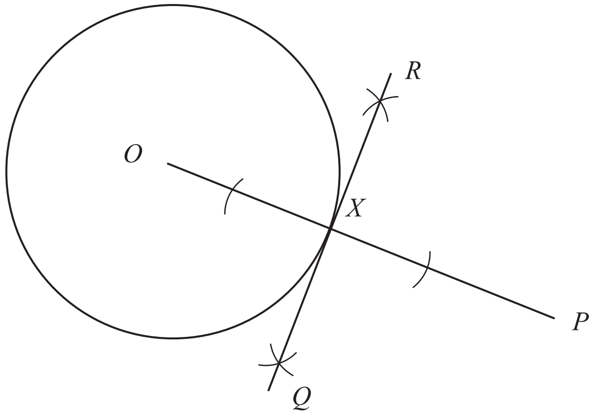


Step 1: Draw the line OX and mark a point P on OX produced.



Step 2: Using the pair of compasses, construct a perpendicular to OP through X.
To do this, use the knowledge on constructing a perpendicular to a straight line segment through a point on it.

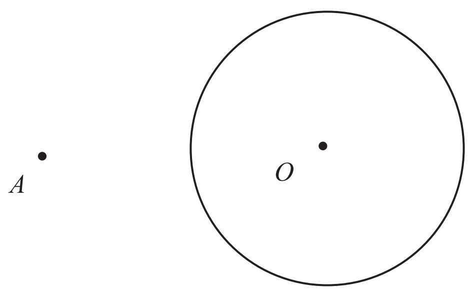
Step 3: Name that perpendicular as RQ.



RQ is the tangent drawn to the circle through X.

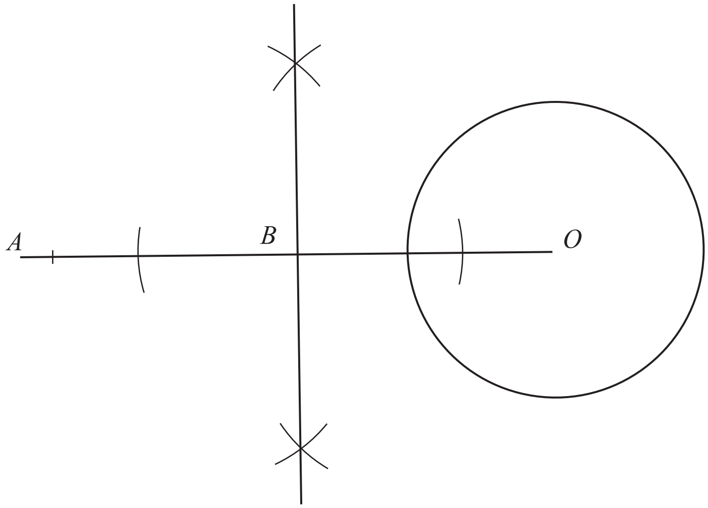
Construction of a tangent to a circle from an external point

Let O be the centre of the circle and A a point external to the circle.



To construct this, let us use the theorem, "The tangents drawn to a circle from an external point (exterior) are equal in length."

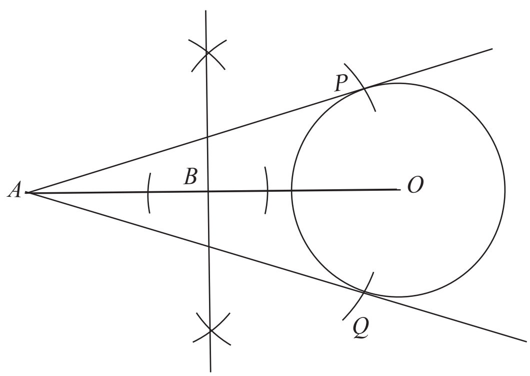
Step 1: Join OA. Construct the perpendicular bisector of the straight line segment OA. Name the point of intersection of OA and its perpendicular bisector as B. To do this construction, use the knowledge on constructing the perpendicular bisector of a straight line segment.



Step 2: Taking B as the centre and BO (or BA) as the radius, draw two arcs which intersect the circle.

Step 3: Name the two points of intersection as P and Q.

Step 4: Draw the lines AP and AQ.



AP and AQ are the tangents drawn to the circle with centre O from the point A. Using the protractor we can establish this by showing that the angles $A\hat{P}O$ and $A\hat{Q}O$ are 90°.

Exercise 23.3

1. Construct a circle of radius 3 cm. Mark a point on the circle and name it A. Construct a tangent to the circle through the point A.

2. (i) Construct a circle of radius 3.5cm and name its centre O. Mark a point on the circle and name it P. Construct a tangent to the circle through the point P.
(ii) Mark a point Q on the tangent such that $PQ = 5$cm.
(iii) Measure and write down the length of OQ.
(iv) Find the length of OQ using Pythagoras's theorem and check if your answer is correct.

3. (i) Construct the equilateral triangle ABC of side length 5 cm.
(ii) Construct the circle which touches AB at B and passes through the point C.
(iii) Measure the radius of this circle and write it down.

4. (i) Construct the circle with centre O and radius 2.8 cm.
(ii) Mark a point A on the circle and join OA. Mark the point B on OA produced such that $OB = 5$ cm.
(iii) Construct tangents to the circle from the point B.
(iv) Measure and write down the lengths of the tangents.

5. (i) Construct the triangle ABC such that $AB = 5$ cm, $AC = 3$ cm and $B\hat{A}C = 90^{o}$.
(ii) Construct the circumcircle of ABC.
(iii) Construct a tangent to the circle through the point A.
(iv) Name the point of intersection of the tangent drawn through A and BC produced as P.
(v) Construct another tangent to the circle from the point P.

6. (i) Construct a triangle KLM such that $KL = 9$ cm, $K\hat{L}M = 90^{o}$ and $LM = 4$ cm.
(ii) Construct the angle bisector of $K\hat{M}L$. Name the point where it meets KL as O.
(iii) Construct a circle taking O as the centre and OL as the radius.
(iv) Mark a point T on KM such that $ML = MT$.
(v) Find the magnitude of $O\hat{T}M$.
(vi) Draw another tangent to the circle from the point K.

Miscellaneous Exercise

1. (i) Construct the triangle ABC such that $AB = 6$cm, $A\hat{B}C = 45^{\circ}$ and $BC = 4$ cm.

(ii) Construct a line through A parallel to BC.

(iii) Construct the circle which has its centre on this parallel line and which passes through the points A and B.

2. (i) Construct the triangle PQR where $PQ = 7$cm, $P\hat{Q}R = 120^{\circ}$ and $QR = 4.5$ cm.

(ii) Locate the point S such that $PQRS$ is a parallelogram.

(iii) Draw the diagonal QS.

(iv) Construct the circumcircle of triangle PQS.

(v) Construct the incircle of triangle QRS.

3. Construct the triangle PQR such that $PQ = 4.8$ cm, $P\hat{Q}R = 90^{\circ}$ and $QR = 6.5$ cm. Construct a circle which touches PQ at P and also touches QR.

24 Sets

By studying this lesson you will be able to

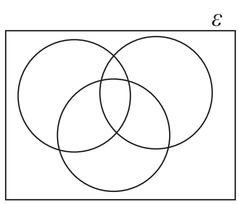
- identify the regions in a Venn diagram,
- express the regions in a Venn diagram using set notation,
- solve problems using Venn diagrams involving three sets.

Venn diagrams

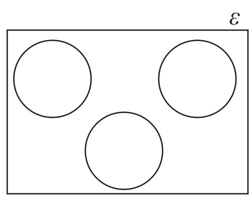
You have learnt in Grade 10 to identify regions in Venn diagrams which involve two sets and to express shaded regions in Venn diagrams using set notation. You can represent three subsets of the universal set in a Venn diagram too. Let us now consider how this is done.

The following are different ways in which three nonempty subsets of a universal set can be represented in a Venn diagram. The first figure illustrates the most general form.

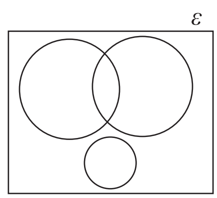
(i)



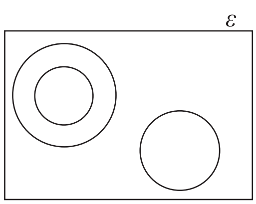
(ii)



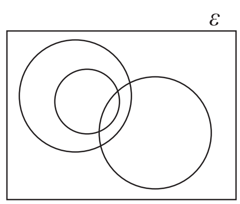
(iii)



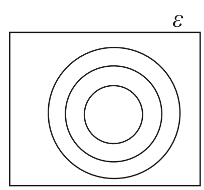
(iv)



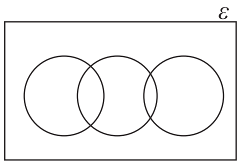
(v)



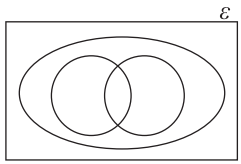
(vi)



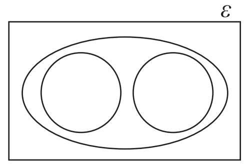
(vii)



(viii)

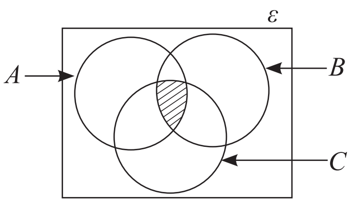


(ix)

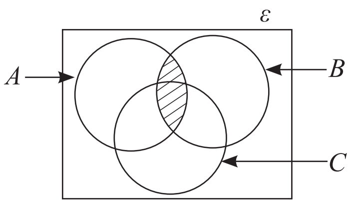


24.1 Expressing a subset denoted by a shaded region in a Venn diagram using set notation

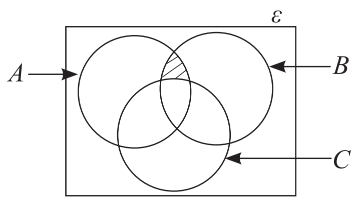
Let A, B and C be three nonempty subsets of a universal set. Several cases of shaded regions in a Venn diagram which have been expressed using set notation are given below.



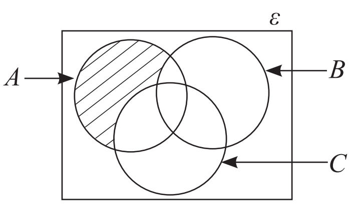
$A \cap B \cap C$



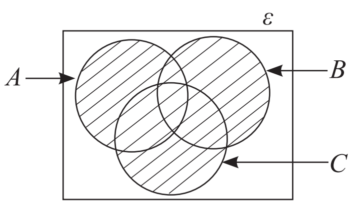
$A \cap B$



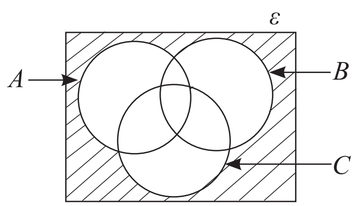
$(A \cap B) \cap C'$



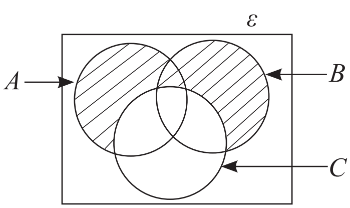
$A \cap (B \cup C)'$



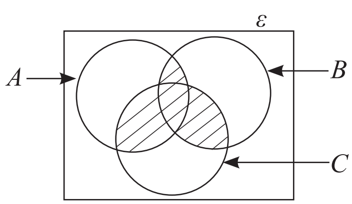
$A \cup B \cup C$



$(A \cup B \cup C)'$



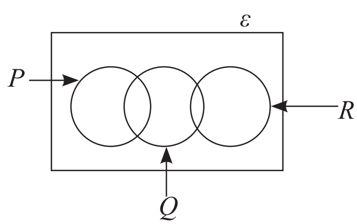
$(A \cup B) \cap C'$



$(A \cap B) \cup (B \cap C) \cup (C \cap A)$

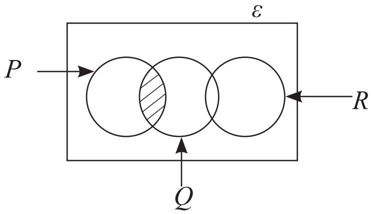
Example 1

For each of the following cases, shade the region representing the given set in a copy of the Venn diagram provided here.

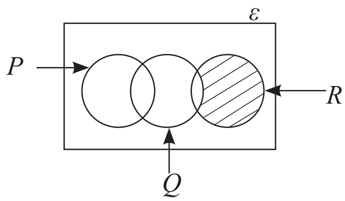


(i) $P \cap Q$ (ii) $(P \cup Q)' \cap R$ (iii) $(P \cup R)' \cap Q$
(iv) $(P \cup Q \cup R)'$

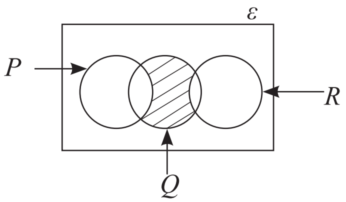
(i) $P \cap Q$



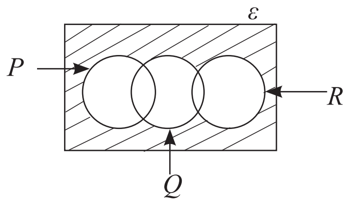
(ii) $(P \cup Q)' \cap R$



(iii) $(P \cup R)' \cap Q$

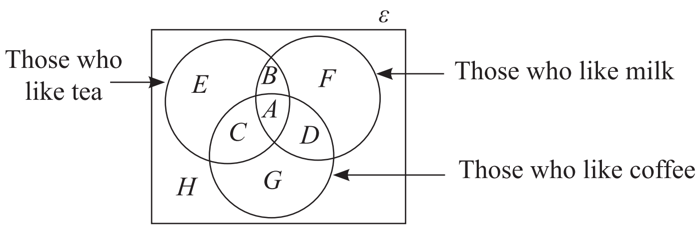


(iv) $(P \cup Q \cup R)'$



Next we will consider how to describe the nature of the elements that belong to a particular region in a Venn diagram in words. It is easier to understand this if we consider an example.

The following Venn diagram provides information on what a group of students like to drink.



The regions denoted by the capital letters of the English alphabet in the above Venn diagram can be described in words as follows.

A - Those who like to drink tea, milk and coffee

B - Those who like to drink only tea and milk. That is, those who like to drink tea and milk but do not like to drink coffee

C - Those who like to drink only tea and coffee

D - Those who like to drink only milk and coffee

E - Those who like to drink only tea

F - Those who like to drink only milk

G - Those who like to drink only coffee

H - Those who do not like to drink any of these three

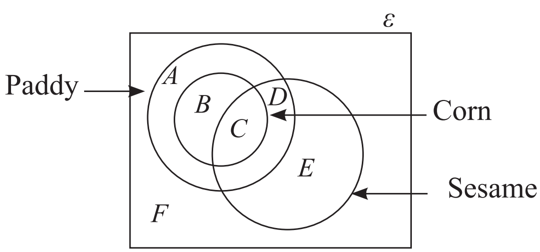
Moreover, regions that are obtained by combining two or more of the above regions can also be described in words, most often in a simple way.

A and *B* – Those who like to drink tea and milk
B and *C* and *D* – Those who like to dring exactly two of these three types
A and *B* and *C* and *D* – Those who like to drink at least two of these three types
A and *B* and *C* and *E* – Those who like to drink tea
E and *F* and *G* – Those who like to drink exactly one of these three types

Example 2

The following Venn diagram provides information on the types of crops that a group of farmers cultivate. Describe each of the subsets which are denoted by the capital letters as well as the subsets denoted by the following composite regions.

(i) *B* and *C*
(ii) *C* and *D*
(iii) *A* and *D* and *E*

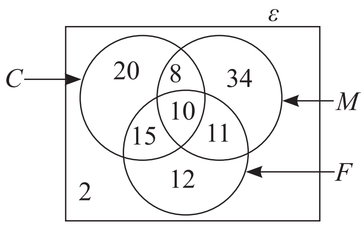


A – Farmers who cultivate only paddy
B – Farmers who cultivate only paddy and corn
C – Farmers who cultivate paddy, corn and sesame
D – Farmers who cultivate paddy and sesame but not corn
E – Farmers who cultivate only sesame
F – Farmers who do not cultivate any of these three crops
B and *C* – Farmers who cultivate corn
C and *D* – Farmers who cultivate paddy and sesame
A and *D* and *E* –Farmers who cultivate at least one crop but do not cultivate corn

Example 3

Let ε = {Families living in a Housing Scheme}
C = {Families that own a car}
M = {Families that own a motorcycle}
F = {Families that own a bicycle}

These sets have been represented in the following Venn diagram. The numbers represent the number of elements in each subset.
Answer the following questions by considering the Venn diagram.



(i) How many families own a car?
(ii) How many families own a motorcycle but neither a car nor a bicycle?
(iii) How many families do not own a bicycle?
(iv) How many families own exactly two types of vehicles?
(v) How many families own at least two types of vehicles?
(vi) How many families own exactly one of these three types of vehicles?

(i) The set of households which have cars is denoted by C. All these households need to be considered. Therefore, the number of households which have cars is $20 + 8 + 10 + 15 = 53$.

(ii) The set of households that have motor cycles is denoted by M. The households which have only motorcycles are those which have motorcycles but not cars or bicycles. Therefore, to obtain the answer, from the households which have motorcycles, those which also have either cars or bicycles have to be removed. Hence, the number of households which have only motorcycles is 34.

(iii) The number of households which do not have bicycles is obtained by removing from all the households in the scheme, those which have bicycles. In other words, these households are those which have only cars, only motor cycles, only cars and motorcycles or those which do not have any of these three types of vehicles. This number is, $20 + 8 + 34 + 2 = 64$

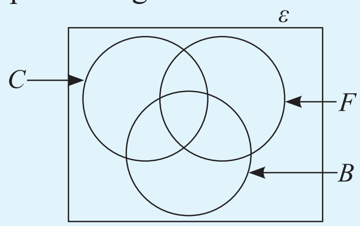
(iv) The households which have only two types of vehicles are those which have only cars and motorcycles or only cars and bicycles or only motorcycles and bicycles. This number is $15 + 8 + 11 = 34$.

(v) The households which have at least two types of vehicles are those which have either two types of vehicles or all three types of vehicles. This number is $15 + 8 + 11 + 10 = 44$.

(vi) The households which have only one type of vehicle are those which have only cars or only motorcycles or only bicycles. This number is $20 + 34 + 12 = 66$.

Exercise 24.1

1. A Venn diagram prepared based on the types of sports that each student in a group likes is given below.

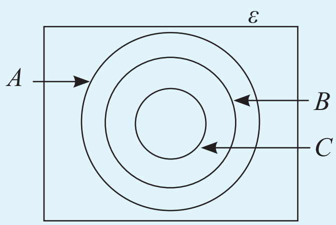


C = {Those who like cricket}
F = {Those who like football}
B = {Those who like volleyball}

By using the above Venn diagram model, shade the region that denotes each of the following sets expressed in set notation and describe it in words. Use a different Venn diagram for each of the parts (i), (ii), (iii) and (iv).

(i) $B \cap C \cap F$ (ii) $(C \cap F) \cap B'$ (iii) $(B \cup C)' \cup F$ (iv) $(B \cup C \cup F)'$

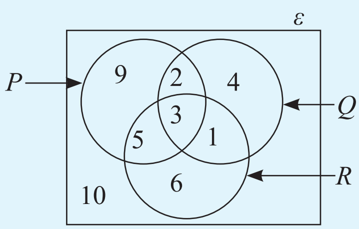
2.



By using the given Venn diagram model, shade the region that denotes each of the following sets expressed in set notation and describe it in words. Use a different Venn diagram for each of the parts (i), (ii), (iii) and (iv).

(i) $A \cap B \cap C$ (ii) $B \cap C'$
(iii) $A \cap (B \cup C)'$ (iv) $(A \cup B \cup C)'$

3.



Determine the following based on the given Venn diagram.
(i) $n\,(P \cap Q \cap R)$ (ii) $n\,(Q \cup R)'$
(iii) $n\,[(P \cap Q) \cap R')]$ (iv) $n\,[(Q \cup R)' \cap P]$
(v) $n\,(P \cup Q \cup R)'$
The numbers given in the Venn dragram are the number of elements in the corresponding region.

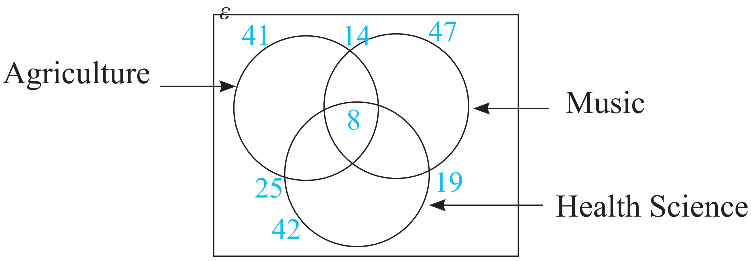
24.2 More on problems related to sets

Let us see how to solve problems related to sets by considering a few examples.

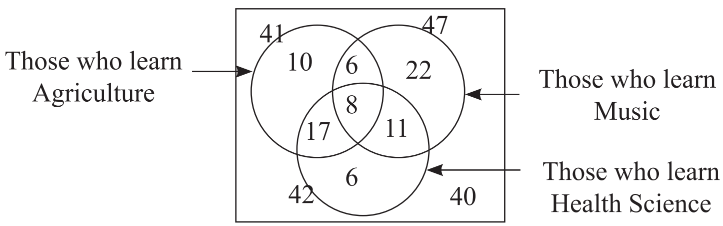
Example 1

From a group of 120 students, 41 learn agriculture, 47 learn music and 42 learn health science. 14 students learn agriculture and music, 19 learn music and health science, 25 learn agriculture and health science and 8 learn all three subjects. Represent this information in a Venn diagram and thereby determine the following.

(i) The number of students who learn only agriculture
(ii) The number of students who learn only one of these subjects
(iii) The number of students who learn at least two of these subjects
(iv) The number of students who do not learn any one of these three subjects



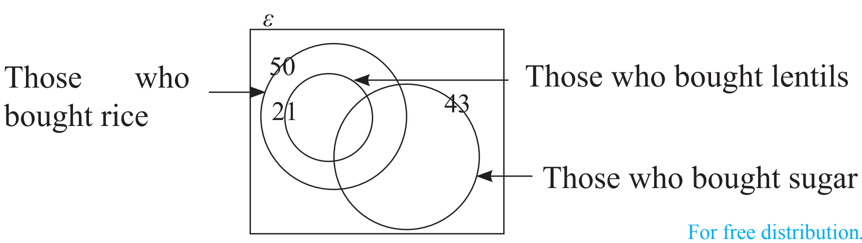
Let us find the number of elements in the remaining regions using the given information.



(i) 10
(ii) 10 + 22 + **6** **= 38**
(iii) 17 + **6** + 11 + 8 = 42
(iv) 40

Example 2

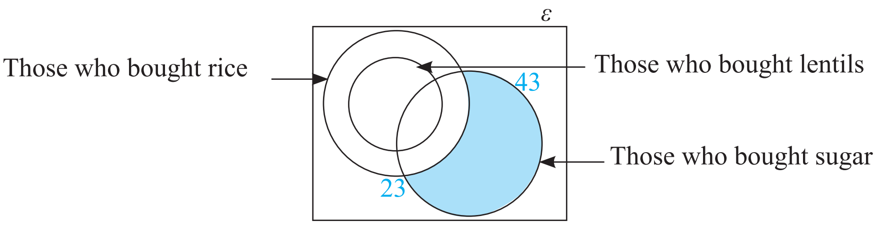
According to the information gathered on the customers who visited a particular grocery store during a certain hour on a certain day, 50 bought rice, 21 bought lentils and 43 bought sugar. Moreover, everyone who bought lentils also bought rice. The given Venn diagram provides this and additional information.



(i) Twenty three customers bought rice and sugar. How many bought only sugar?
(ii) Twelve customers bought all three items. How many bought only rice and lentils?
(iii) How many bought only rice?
(iv) If 90 people came to the grocery store during that hour, how many came to buy other items?

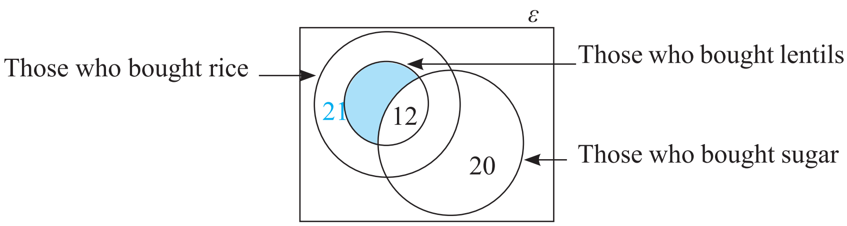
Answers
Let us find the number of elements belonging to each region using the given information.

(i)



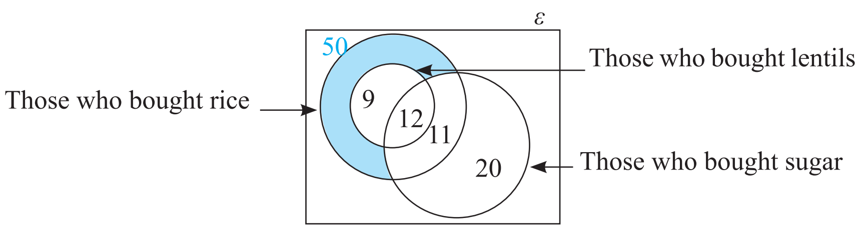
The number of customers who bought only sugar is = 43 – 23 = 20.

(ii)



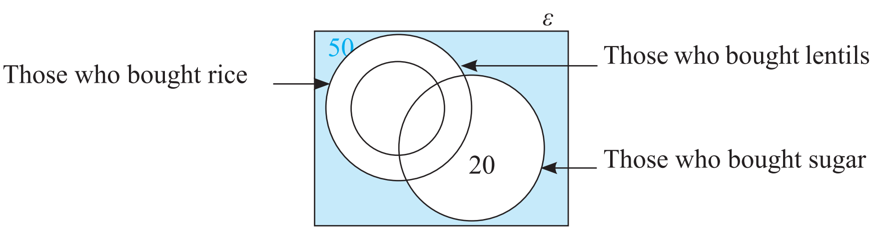
The number of customers who bought only rice and lentils is 21 – 12 = 9.

(iii)



The number of customers who bought only rice is 50 – 9 – 12 – 11 = 18.

(iv)



The number of customers who bought items other than rice, lentils and sugar
= 90 – 70 = 20.

Exercise 24.2

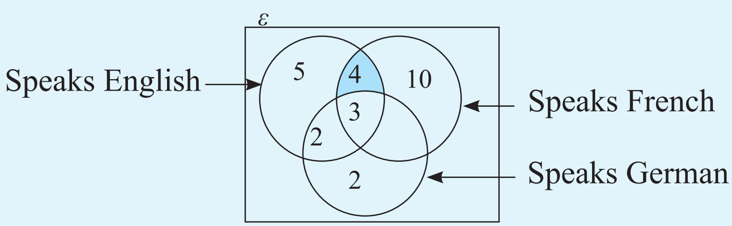
1. Information on 20 customers who bought items from a certain stationary shop is as follows. There were 8 who bought pencils, 11 who bought pens and 13 who bought books. Of the 6 who bought pencils and books, 4 did not buy pens. There were 3 who bought pencils and pens and 3 who bought only pens. Represent this information in a Venn diagram and determine the following.

(i) How many did not buy any of these three types of items?
(ii) How many did not buy pens?
(iii) What percentage of all the customers bought at least 2 of these 3 types?

2. The following information on the newspapers *A, B* and *C* that are purchased by a village community was obtained through a survey. 50% buy *A*, 67% buy *B* and 55% buy *C*. 10% buy only *A* and *B*, 15% buy only *A* and 5% buy *A* and *C* but not *B*. 17% do not buy *A* but buy *B* and *C*. Represent this information in a Venn diagram and determine the following.

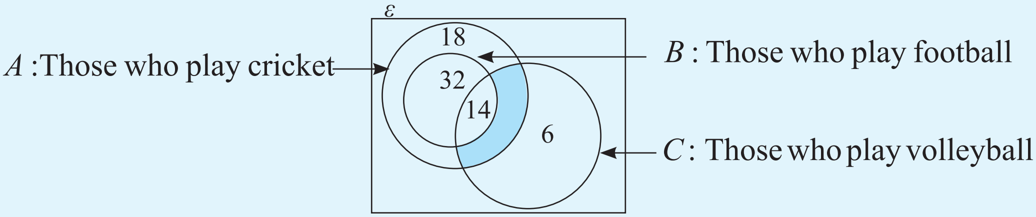
(i) The percentage that buy all three newspapers.
(ii) The percentage that buy *C* but not *A*.
(iii) The percentage that buy only two of these three types.

3. The following Venn diagram has been drawn with the information that was noted down on the languages that a group of tourists who visited Sigiriya could speak.



(i) How many can speak English?
(ii) If 12 people in total can speak German, how many can speak French and German only?
(iii) Describe in words the set represented by the shaded region.
(iv) All those who could speak English stayed on with the guide who provided commentaries in English while the rest were sent with a guide who was fluent in both French and German. How many went with this guide?

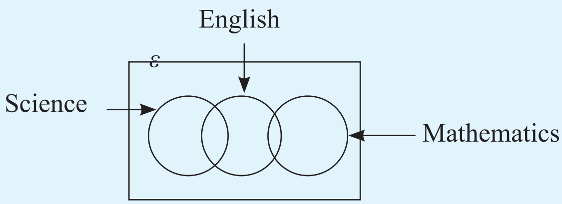
4. All the students who receive training at a certain sports academy participate in at least one of the three sports cricket, football and volley ball. The Venn diagram provides information on these students.



(i) How many students participate in all three sports?
(ii) How many students play only cricket?
(iii) Describe the group of students who are represented by the shaded region and express this set using set notation.
(iv) If 25 play volleyball, how many students belong to the shaded region?

5. Each year, 400 students are enrolled in a National College of Education. The three subjects, namely mathematics, science and physical education which are taught there are taught in both, Sinhala and English mediums.

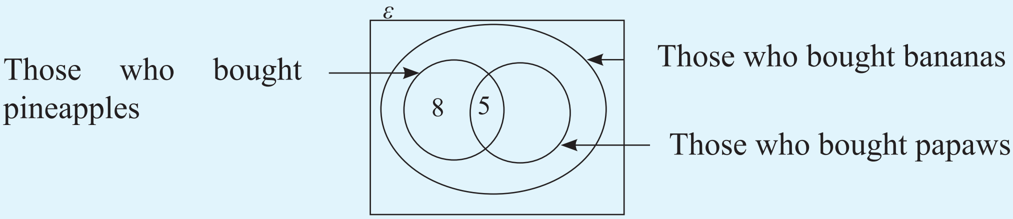
(a) Complete the following Venn diagram by marking the given information in the correct region.



(i) There are 140 students who study the subject science. 100 of these students study in the Sinhala medium.
(ii) 40 students study mathematics in the English medium.
(iii) 110 students study in the English medium
(iv) The total number of students who study mathematics is 175.

(b) (i) How many students study physical education in the English medium?
(ii) How many students study science in the English medium?
(iii) How many students study mathematics in the Sinhala medium?
(iv) Find the probability of a student selected at random from those who enrolled that year being a student who studies physical education in the Sinhala medium.

6. Information on the types of fruits that were purchased by the customers who arrived at a grocery store one day is given in the following Venn diagram. On that day, all who bought either pineapples or papaws also purchased bananas.



(i) How many bought pineapples?
(ii) If 12 people bought papaws, how many bought only papaws?
(iii) If 40 people bought bananas, how many bought only bananas?
(iv) If there were 10 other customers who purchased fruits but did not buy any one of the given three types, how many came to the grocery store that day to buy fruits?
(v) How many bought only two of these three types of fruits?
(vi) If a person was selected at random from those who came to buy fruits, what is the probability that the person purchased all three of the given types of fruits?

25 Probability

By studying this lesson you will be able to,

- solve problems involving the events of a random experiment occurring in two stages using
  (i) a grid (Cartesian plane)
  (ii) a tree diagram

Do the following exercise to recall what had been learnt in Grade 10.

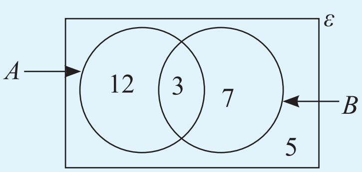
Review Exercise

1. A is an event in the sample space S of a random experiment with equally likely outcomes. If $n(A) = 23$ and $n(S) = 50$, find
   (i) $P(A)$
   (ii) $P(A')$.

2. The sample space S of a random experiment is $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Assuming that the outcomes of this experiment are equally likely, answer the questions given below.

   (i) A is a simple event of the above experiment. Write down all the events that A could be.
   (ii) For each of the above events, find $P(A)$.
   (iii) B is a compound event of the above experiment consisting of 4 elements. Write one event that B could be.
   (iv) Find $P(B)$ and $P(B')$.
   (v) X is another event with $P(X) = 0.5$ Write two events that X could be.

3. The following Venn diagram provides information on the number of elements in the subsets relevant to two events of a random experiment.
   (a) Determine the following



(i) $n(\varepsilon)$ (ii) $P(A)$ (iii) $P(B)$
(iv) $P(A \cap B)$ (v) $P(A \cup B)$ (vi) $P(A \cap B')$
(vii) $P(A' \cap B)$ (viii) $P(A \cup B)'$

4. Three identical cards numbered from 1 to 3 are in a bag. One card is drawn out at random and the number on it is checked to see if it is odd or even, and then the card is replaced in the bag. A card is drawn out again at random from the bag and the number is similarly checked to see whether it is odd or even.

   (i) If the sample space is denoted by S, write S as a set and write down $n(S)$.
   (ii) If A is the event of both numbers drawn being even, write A as a set and write down $n(A)$.
   (iii) Hence find $P(A)$.
   (iv) Represent S on a grid.
   (v) If B is the event of drawing exactly one even number, then square the points on the grid that belong to B and find $P(B)$.
   (vi) Represent S in a tree diagram and find the probability of drawing at least one even number.

25.1 Independent and Dependent Events

(i) Independent events

We learnt in Grade 10 that if the occurrence of an event does not depend on the occurrence of another event, then the two events are independent. If A and B are independent we know that $P(A \cap B) = P(A)P(B)$. An example of this is given below.

Let us consider the random experiment of flipping two coins simultaneously and noting the sides that are face up. It is clear that the side which is face up on one coin has no influence on the side which is face up on the other coin. Therefore the side which is face up on one coin is independent of the side which is face up on the other coin.

(ii) Dependent events

If the occurrence of an event depends on the occurrence of another event, then they are called dependent events. That is, the occurrence of one event changes the probability of the other event.

Deepen your understanding on dependent events further by studying the following examples.

a. The probability of a team winning a cricket match depends on whether the best bowler of the team plays in the match or not. Therefore the two events of the best bowler playing and the team winning are dependent events.

b. An animal is selected at random from a cattle pound in which there are cows and bulls. If the selected animal is a cow, milk can be obtained from it and if it is a bull, then milk definitely cannot be obtained from it. Therefore the event of selecting a cow and the event of obtaining milk are dependent events.

c. A bag contains 7 white balls and 3 black balls which are identical in size and shape. Let us consider the experiment of randomly drawing out a ball and noting its colour, and then, without replacing the first ball, randomly drawing out another ball, and noting its colour. As the first ball is not replaced before the second ball is drawn, the number of balls in the bag when the second one is drawn out is not 10 but 9, and the number of balls of a particular colour remaining in the bag after the first ball is drawn depends on the colour of the first ball that is drawn.

The probability of the second ball being a white ball if the first ball was white $= \frac{6}{9} = \frac{2}{3}$

The probability of the second ball being a white ball if the first ball was not white $= \frac{7}{9}$

As these two probabilities are not equal, the probability of the second ball being white is dependent on the probability of the first ball being white.

25.2 Solving problems using a grid

When a random experiment occurs in two stages, the events of the two stages may be independent or dependent. In Grade 10 we learnt how to solve problems when the events of the two stages are independent. To review what was learnt, let us consider the following example.

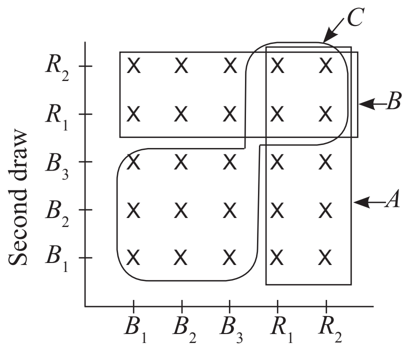
Example 1

In a bag there are 3 blue balls and 2 red balls of the same shape and size. A ball is randomly drawn out and the colour is noted. After replacing it, a ball is randomly drawn out again and its colour is also noted.

(i) Represent the sample space of this random experiment on a grid.
(ii) Find the probability of each of the following events using the grid.

(a) The first ball being red
(b) The second ball being red
(c) Both balls being red
(d) Both balls being the same colour
(e) At least one ball being red

(i) We have learnt earlier that a grid can be used to solve problems on probability only if all the outcomes of the random experiment are equally likely. Since all the balls are equal in size and shape, the probability of a ball being drawn out is the same for all the balls. Therefore the sample space can be represented on a grid and the required probabilities can be found. Let us name the three blue balls B_1, B_2 and B_3 and the two red balls R_1 and R_2.



The sample space consists of all the points that are marked by taking the horizontal axis to represent the outcomes of the first draw and the vertical axis to represent the outcomes of the second draw.

Since the second ball is taken out after replacing the first ball, the two events are independent of each other.

The probability of an event is found using a grid, by dividing the number of points relevant to the given event by the total number of points in the sample space.

(ii) The event of drawing a red ball on the first occasion is denoted by A on the grid. 10 points from the sample space belong to A. The whole sample space consists of 25 points.

$\therefore$ the probability of the first ball being red $= \dfrac{\text{number of points in } A}{\text{number of points in the sample space}}$

$$= \frac{10}{25} = \frac{2}{5}$$

(b) The event of the second ball being red is denoted by B on the grid.

Accordingly,

the probability of the second ball being red $= \dfrac{\text{number of points in } B}{\text{number of points in the sample space}}$

$$= \frac{10}{25} = \frac{2}{5}$$

(c) The event of both balls being red is the set of points which are common to both A and B. There are 4 points in this set.

$$\therefore \text{ the probability of both balls being red} = \frac{\text{number of points common to both } A \text{ and } B}{\text{number of points in the sample space}}$$

$$= \frac{4}{25}$$

(d) For both balls to be the same colour, they should both be blue or both be red. The set of points relevant to this event is denoted by C. There are 13 points in this set.

$$\therefore \left.\begin{array}{l}\text{the probability of both balls being}\\ \text{the same colour}\end{array}\right\} = \frac{\text{number of points in } C}{\text{number of points in the sample space}}$$

$$= \frac{13}{25}$$

(e) At least one ball being red means that one ball is red or both balls are red. This means all the points in both A and B. There are 16 points in total.

$$\therefore \text{ the probability of at least one ball being red} = \frac{16}{25}$$

Now let us consider an example of a random experiment consisting of two stages, where the events are dependent.

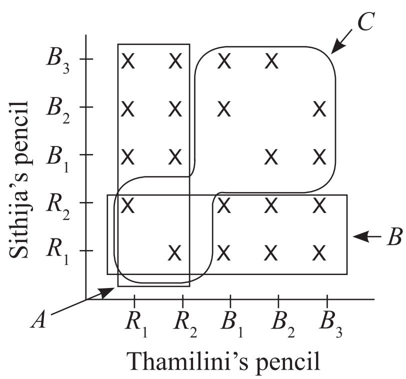
Example 2

In Sithija's pencil box there are 2 red pencils and 3 blue pencils of the same size and shape. Sithija randomly draws one pencil out and gives it to his friend Thamilini. Then Sithija randomly draws another pencil out for himself.

(i) Write the sample space in terms of the outcomes and then represent it on a grid.
(ii) Find the probability of each of the following events using the grid.
  (a) Drawing a red pencil for Thamilini
  (b) Sithija drawing a red pencil for himself
  (c) Both getting pencils of the same colour
  (d) Only Thamilini getting a red pencil

(i) Let us name the three blue pencils B_1, B_2, B_3 and the two red pencils R_1, R_2.The pencil given to Thamilini is one of R_1, R_2, B_1, B_2, B_3 and the pencil Sithija drew for himself is also one of these. However since Sithija cannot have the pencil which was given to Thamilini, the events (R_1, R_1), (R_2, R_2), (B_1, B_1), (B_2, B_2) and (B_3, B_3) cannot occur. Apart from these 5 points, the remaining 20 belong to the sample space. Therefore the sample space is the set

$\{(R_1, R_2), (R_1, B_1), (R_1, B_2), (R_1, B_3), (R_2, R_1), (R_2, B_1)...\}$. This sample space can be represented on a grid as follows.



(a) The 8 points corresponding to Thamilini receiving a red pencil is marked as A .

$\therefore$ probability of Thamilini receiving a red pencil $= \frac{8}{20} = \frac{2}{5}$

(b) The 8 points corresponding to Sithija getting a red pencil is marked as B.

$\therefore$ probability of Sithija getting a red pencil $= \frac{8}{20} = \frac{2}{5}$

(c) The set of points corresponding to both getting pencils of the same colour is marked as C. This is the event of either both getting red pencils or both getting blue pencils. C also has 8 points.

$\therefore$ probability of both getting pencils of the same colour $= \frac{8}{20} = \frac{2}{5}$

(d) For only Thamilini to get a red pencil, Sithija has to get a blue pencil while Thamilini gets a red pencil. There are 6 points corresponding to this.

$\therefore$ probability of only Thamilini getting a red pencil $= \frac{6}{20} = \frac{3}{10}$

Exercise 25.1

1. In a box, there are 2 white balls and 4 red balls of the same shape and size. One ball is drawn out randomly from this box and its colour is noted.

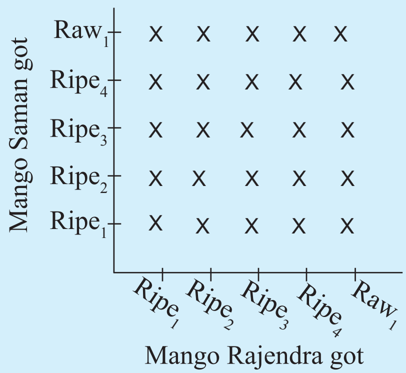
(a) Write down the sample space S of all possible equally likely outcomes.

(b) If the ball is put back into the box and a ball is randomly drawn out again and its colour is also noted, draw the sample space of the equally likely outcomes of this experiment on a grid.

(c) If the ball is not put back into the box and another ball is randomly drawn out and its colour is noted, draw the sample space of this experiment on a grid.

(d) For the experiments described in (b) and (c) above, find separately, the probability of the two balls taken out being of the same colour.

2. In a bag there are 4 ripe mangoes and 1 raw mango, all of the same size and shape. Saman randomly takes one mango from this bag and gives it to his friend Rajendra. After that Saman randomly takes out a mango for himself. A grid with the sample space prepared by Saman is given below.



(a) There is an error in the above sample space. Draw the correct sample space on another grid.

(b) Using the correct sample space, find the probability of each of the following events.

(i) Both getting ripe mangoes
(ii) Only Rajendra getting a ripe mango
(iii) Only one person getting a ripe mango

(c) Rajendra says that at least one person is sure to get a ripe mango. Explain with reasons whether this is true or not.

3. Sarath who was preparing for a journey, randomly took out two shirts (one after the other) from a box containing 4 white shirts and 3 black shirts.

(a) Represent the sample space on a grid by denoting the white shirts in the box by W_1, W_2, W_3, W_4 and the black shirts by B_1, B_2, B_3.

(b) Using the grid, find the probability of each of the following events.

(i) Both shirts being white
(ii) Only one shirt being white
(iii) At least one shirt being white

4. In a dish there were 3 milk toffees, 2 orange flavoured toffees and 1 tamarind flavoured toffee which were all the same size and shape. Sandaru randomly took one of these toffees and tasted it. Later she randomly took another one and gave it to her friend Jessie.

(a) By taking into consideration the flavours of the toffees, represent the relevant sample space of equally likely outcomes on a grid.

(b) Using the grid, find the probability of each of the following events.

(i) Both getting toffees of the same flavour.
(ii) Only one person getting a milk toffee.
(iii) Jessie getting a tamarind flavoured toffee.

25.3 Solving problems using tree diagrams

A tree diagram can be used to find the probabilities of events related to a random experiment having many stages. In this lesson we will consider only random experiments having two stages. Let us study this by considering the following examples.

In Grade 10 you learnt to find probabilities when the events were independent. Let us review what you learnt earlier.

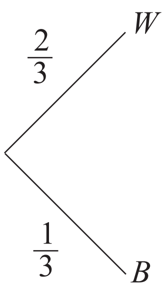
Example 1

In a bag there are two white balls and one black ball of the same size and shape. One ball is randomly drawn out and its colour is noted. After replacing it, a ball is randomly drawn out again and its colour is also noted.

(i) Represent the sample space of this random experiment in a tree diagram.
(ii) Find the probability of each of the following events using the tree diagram.
(a) Drawing a white ball on both occasions
(b) Drawing a white ball on the first occasion
(c) Drawing only one white ball
(d) Drawing at least one white ball

(i) Let us denote the event of drawing a white ball by W and drawing a black ball by B. As the outcomes are equally likely, the probability of drawing a white ball out the first time is $\frac{2}{3}$ and a back ball out the first time is $\frac{1}{3}$. In that part of the tree diagram which represents the first draw, let us indicate the probability of each of these two events on the relevant branches.

First draw



The sum of the probabilities on the two branches $= \frac{2}{3} + \frac{1}{3}$

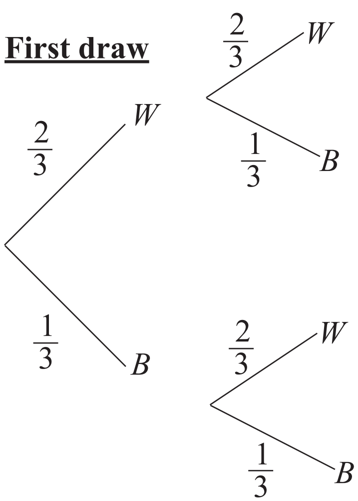
$= 1$

Note: The sum of the probabilities on the branches which originate from a single point is 1. This occurs at every stage.

Now let us extend the above tree diagram to indicate the probabilities of the second stage.

Second draw

First draw



As the second ball is drawn out after replacing the first ball, the number of balls in the bag when the second ball is drawn out is the same as the number that was in the bag initially. Therefore the probabilities of drawing a white ball and drawing a black ball on the second occasion are the same as those on the first occasion. These probabilities are marked on the appropriate branches.

Notice that the sum of the probabilities on the branches that originate from a point, all add up to

(ii) When both occasions are considered there are 4 possible outcomes.

Outcome	**Probability**	
(W, W)	$\frac{2}{3} \times \frac{2}{3}$	$\frac{4}{9}$
(W, B)	$\frac{2}{3} \times \frac{1}{3}$	$\frac{2}{9}$
(B, W)	$\frac{1}{3} \times \frac{2}{3}$	$\frac{2}{9}$
(B, B)	$\frac{1}{3} \times \frac{1}{3}$	$\frac{1}{9}$

As an example, (W,W) represents the event of both balls drawn being white. Its probability is $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$. The reason why these two can be multiplied together is because they are independent events. The four events (W, W), (W, B), (B, W) and (B, B) are mutually exclusive events. The reason for this is because, no two of these four events can occur together. The required probabilities can be calculated as follows.

(a)Probability of drawing a white ball on the first occasion and a white ball on the second occasion too

$$= P(W, W)$$
$$= \frac{4}{9} \text{ (from the table)}$$

(b) Probability of drawing a white ball on the first occasion $= P(W, W) + P(W, B)$

$$= \frac{4}{9} + \frac{2}{9} = \frac{6}{9} = \frac{2}{3}$$

(c) Probability of drawing only one white ball $= P(W, B) + P(B, W)$

$$= \frac{2}{9} + \frac{2}{9} = \frac{4}{9}$$

(d) Probability of drawing at least one white ball $= P(W, W) + P(W, B) + P(B, W)$

$$= \frac{4}{9} + \frac{2}{9} + \frac{2}{2} = \frac{8}{9}$$

Note: The answer in (d) can be obtained from $1 - P(B, B)$ too.

Now let us consider an example where the events are dependent.

Example 2

In a bag there are 3 red balls and 2 blue balls of the same size and shape. One ball is randomly drawn out and its colour is noted. Without replacing the first ball, a second ball is randomly drawn out and its colour too is noted.

(i) Represent the sample space in a tree diagram.

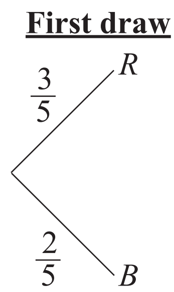
(ii) Using the tree diagram find the probability of each of the following events.

(a) Drawing a red ball on both occasions

(b) Drawing only one red ball

(c) Drawing at least one red ball

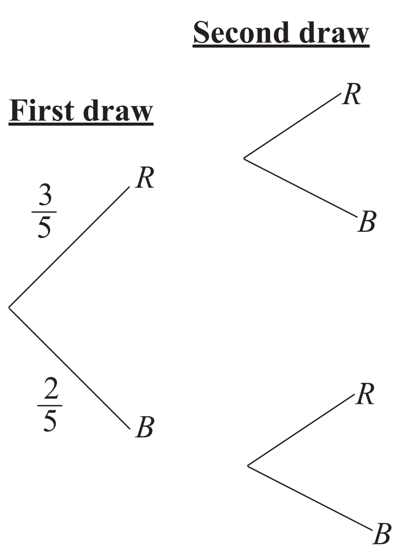
(i) The initial part of the tree diagram is shown below.



Here R represents the event of drawing a red ball while B represents the event of drawing a blue ball. As there are 3 red balls and 2 blue balls in the bag,

$P(R) = \frac{3}{5}, P(B) = \frac{2}{5}$.

Now let us extend the above tree diagram and include the second draw also.



How the probabilities relevant to the second draw were found can be described as follows.

The probabilities on the branches in the second stage are different to those on the branches in the first stage. This is because the probabilities for the second stage have to be found after considering the first stage.

If the first ball drawn is red, then the bag will have 2 red balls and 2 blue balls remaining.

$\therefore$ the probability of the second ball drawn being red $= \frac{2}{4}$

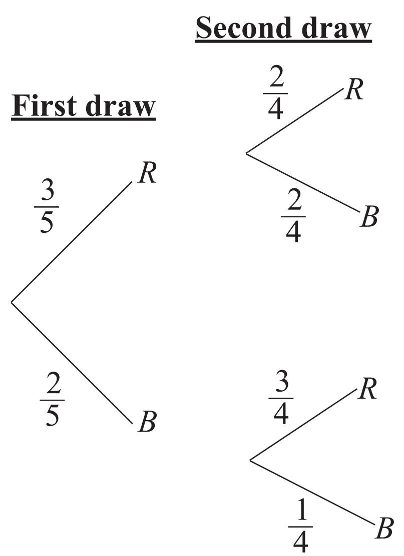
The probability of the second ball drawn being blue $= \frac{2}{4}$

If the first ball drawn is blue, the bag will have 3 red balls and 1 blue ball remaining.

$\therefore$ the probability of the second ball drawn being red $= \frac{3}{4}$

The probability of the second ball drawn being blue $= \frac{1}{4}$

Let us write these probabilities on the relevant branches of the tree diagram, and complete the given outcomes table. Establish that the sum of the probabilities along the four branches is 1.



Outcome	Probability	
(R, R)	$\frac{3}{5} \times \frac{2}{4}$	$\frac{6}{20}$
(R, B)	$\frac{3}{5} \times \frac{2}{4}$	$\frac{6}{20}$
(B, R)	$\frac{2}{5} \times \frac{3}{4}$	$\frac{6}{20}$
(B, B)	$\frac{2}{5} \times \frac{1}{4}$	$\frac{2}{20}$

In the table, the probability of the outcome (R, R) (drawing two red balls) has been found by multiplying the relevant probabilities. But these two probabilities are not independent. This is because the probability of drawing a red ball on the second occasion depends on whether or not a red ball was drawn on the first occasion. Here however, the probability of drawing a red ball on the second occasion has been found by assuming that the first ball drawn is a red ball. Therefore to find the probability of (R, R), the two relevant probabilities can be multiplied.

The events (R, R), (R, B), (B, R), (B, B) in the above table are mutually exclusive. Therefore to find the probability of a certain event using the tree diagram, we need to select the outcomes relevant to the event from the table and add the probabilities of these outcomes.

(a) Probability of drawing two red balls $= P\,(R, R)$

$$= \frac{6}{20} = \frac{3}{10}$$

(b) Probability of drawing only one red ball $= P\,(R, B) + P\,(B, R)$

$$= \frac{6}{20} + \frac{6}{20} = \frac{12}{20} = \frac{3}{5}$$

(c) Probability of drawing at least one red ball $= P\,(R, B) + P\,(B, R) + P\,(R, R)$

$$= \frac{6}{20} + \frac{6}{20} + \frac{6}{20} = \frac{18}{20} = \frac{9}{10}$$

Note: the answer in (c) can be obtained by using $1 - P\,(B, B)$ too.

Exercise 25.2

1. In a box with 10 bulbs of the same type, 3 are known to be faulty. Nimal randomly draws out one bulb from the box and checks whether it is faulty. Without replacing the first bulb he draws out another bulb and checks whether it is faulty.

(i) Represent the sample space of the above random experiment in a tree diagram.

(ii) Nimal states that the events of the first bulb being faulty and the second one being faulty are dependent events. Giving reasons explain whether this statement is true or false.

(iii) Find the probability of each of the following events using the tree diagram.

(a) Both bulbs being faulty

(b) Exactly one bulb being faulty

(c) At least one bulb being faulty

2. The probability of the football player A of a certain team playing in a match is $\frac{3}{4}$. If A plays in the match, then the probability of his team winning is $\frac{5}{8}$ and if he doesn't play, then his team winning or losing is equally likely. This match is either won or lost.

(i) Find the probability of A not playing in the match.

(ii) Find the probability of A's team winning despite him not playing in the match.

(iii) Represent the sample space in a tree diagram, by taking A playing in a match or not as the first stage and the team winning or losing as the second stage.

(iv) Find the probability of A's team winning using the tree diagram.

(v) Giving reasons state whether it is more advantageous or not for the team, if A plays in the match.

3. In a bag there are 4 ripe wood apples and 3 unripe wood apples of the same size and shape. Namali randomly draws out a fruit from the bag. If it is a ripe fruit, she again randomly draws out another one without replacing the first. If the first fruit is unripe, she puts it back into the bag and randomly draws out another fruit.

(i) Represent the sample space of this random experiment in a tree diagram.

(ii) From the following statements made by Namali, with reasons, state which ones are true.

(a) "The first fruit being ripe and the second fruit being ripe are two independent events"

(b) "The first fruit being unripe and the second fruit being unripe are two dependent events"

(iii) Find the probability of each of the following events using the tree diagram.
(a) Both fruits being ripe
(b) Second fruit being ripe
(c) From the two fruits, only one being ripe

4. In Sirimal's cattle pound, there are 5 bulls and 15 cows. In Nadan's cattle pound, there are 2 bulls and 8 cows. Sirimal and Nadan agree to exchange one animal each. After Sirimal randomly selects one animal and sends it to Nadan, Nadan randomly selects one animal and sends it to Sirimal.

(i) Draw the relevant sample space in a tree diagram.
(ii) Using it, find the probability of each of the following events.
(a) There being a reduction in the number of bulls in Sirimal's pound because of the exchange
(b) There being an increase in the number of bulls in Sirimal's pound because of the exchange
(c) There being no difference in the number of bulls and cows in each of the two pounds because of the exchange
(iii) Now suppose they exchange animals in a way different to that mentioned above. Suppose Sirimal and Nadan randomly take one animal from each of their pounds and go to their friend Abdul's house and exchange the cattle there and bring the exchanged cattle back to the pound. Find the probabilities of the events given in (ii) above for this random experiment.

5. X and Y are two drugs given for the same illness which have 90% and 80% effectiveness respectively. If a person does not recover from the illness by using one of the drugs, then the other one is also given. If this too fails, then a surgery is done.

(i) Find the probability of the patient recovering after both drugs are given.
(ii) Find the probability of a patient having to undergo surgery.

6. Information on the clerks and labourers who work in an institute is given in the following table.

Position / Gender	Male	Female	Total
Clerk	5	8	13
Labourer	2	1	3
Total	7	9	16

(i) Find the probability of a person taken randomly from this institute being,
   (a) a labourer
   (b) a clerk
(ii) A clerk and a labourer are randomly taken from this institute.
   (a) Write all the relevant probabilities on the branches of a tree diagram.
   (b) Using the tree diagram, find the probability of at least one of those chosen being male.

7. In a box there are 2 white balls and 1 black ball of the same shape and size. From this box, a ball is drawn out randomly and discarded and then, another is drawn out randomly. Find the probability of at least one of the two drawn balls being white.

8. In box *A* there are 3 blue marbles and 2 red marbles of the same shape and size. In box *B* there are 4 blue marbles and 5 red marbles of the same shape and size as those in box *A*. A marble is drawn out randomly from *A* and placed in *B*. Then a marble is drawn out randomly from *B* and placed in *A*. Find the probability that there has been no change in the colour composition of the marbles in *A*.

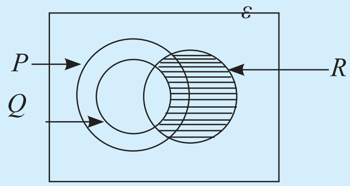
9. In grade 11 of a certain school there are three parallel classes. The number of children in these three classes is in the ratio 2: 2: 3. The teachers who teach mathematics to the three classes are *A, B* and *C* respectively. The principal makes the following statement based on his experience. "90% of the children in the class taught by *A*, 80% of the children in the class taught by *B* and 60% of the children in the class taught by *C* will pass the forthcoming examination."

(i) Find the probability of a randomly chosen child from grade 11 passing the examination, based on the above statement.
(ii) Using the above answer, evaluate the percentage of students who will pass.

Revision Exercise – Term 3

Part I

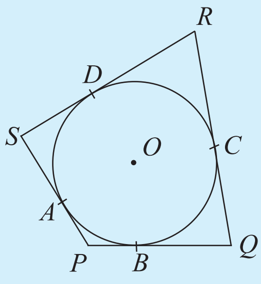
1. Solve the following inequality and represent the solutions on a number line.
$2x + 5 \leq 15$

2. 

Express the shaded region in the Venn diagram using set notation.

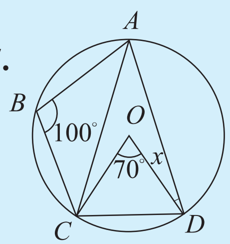
3. The area of the square drawn on the hypotenuse of a right angled isosceles triangle is 64 cm^2. Find the area of a square drawn on one of the other sides.

4. Find p and q if $\begin{pmatrix} -3 & 2 \\ 0 & -5 \end{pmatrix}\begin{pmatrix} p \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ q \end{pmatrix}$.

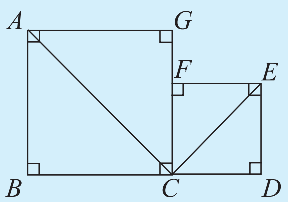
5. 

The tangents drawn to the circle with centre O, at the points A, B, C and D which lie on the circle, meet at the points P, Q, R and S as shown in the figure. If $PQ + SR = 20$ cm, find the perimeter of the quadrilateral $PQRS$.

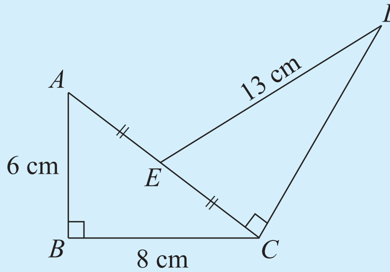
6. A and B are two events of a random experiment such that $P(A) = 0.4$ and $P(A \cup B) = 0.7$. If A and B are independent events, find the value of $P(B)$.

7. 

In the circle with centre O shown in the figure, $C\hat{O}D = 70^\circ$ $C\hat{B}A = 100^\circ$. Find the magnitude of $O\hat{D}A$.

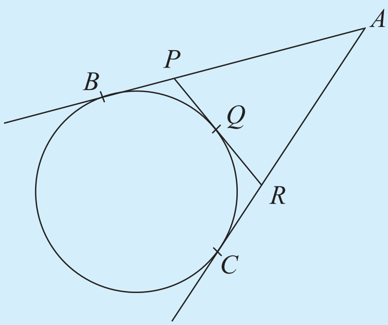
8. 

$ABCG$ and $FCDE$ in the figure are squares. If $AC^2 = 12$ cm^2 and $CE^2 = 6$cm^2, find the area of the whole figure.

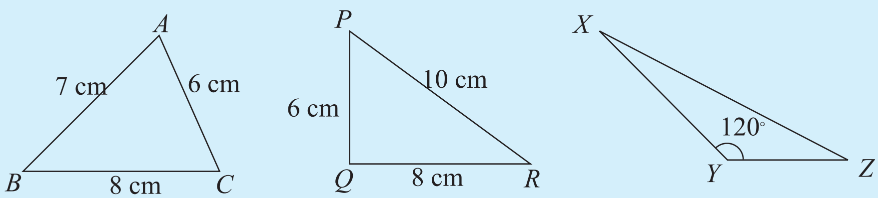
9. 

The triangles ABC and ECD in the figure are right angled triangles. Find the area of the whole figure.

10. Write down the matrix $-2A$ if $A = \begin{pmatrix} -1 & -1 \\ 2 & -2 \end{pmatrix}$

11. 

The tangents drawn from the point A to the circle in the figure are AB and AC. The tangent drawn to the circle at the point Q meets AB and AC at P and R respectively. If the perimeter of the triangle APR is 18 cm, find the length of AB.

12. 

(i) (ii) (iii)

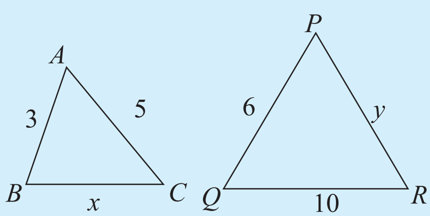
Write down the number of the figure suitable for the blank space.

(a) The circum-centre of the triangle lies on a side of the triangle, in triangles of the form ……………………

(b) The circum-centre of the triangle lies in the exterior of the triangle, in triangles of the form ……………………

(c) The circum-centre of the triangle lies in the interior of the triangle, in triangles of the form ……………………

13. Find the values of x and y in the given equiangular traingles.



14. Represent the integral solutions of the inequality $4x + 3 \geq 8$ on a number line.

15. Write down the coordinates of the turning point of the graph of the function $y = x^2 + 5x + 9$ without drawing the graph.

Part II

1. $A\hat{B}C = 90^\circ$ in the right angled triangle ABC.

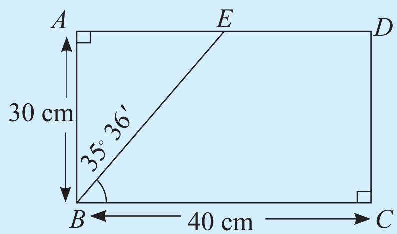
(i) If P is the midpoint of the side BC, then show that $4\,(AP^2 - AB^2) = BC^2$.

(ii) If Q is the midpoint of the side AB, then show that $4\,(CQ^2 - BC^2) = AB^2$.

(iii) Using the results obtained in (i) and (ii) above, deduce that $4\,(AP^2 + CQ^2) = 5\,AC^2$.

(iv) Show using the result obtained in (iii) above, that when the above triangle is an isosceles right angled triangle, then $AP{:}QP = \sqrt{5} : \sqrt{2}$.

2. (a)



$ABCD$ is a rectangle. Using the trigonometric tables

(i) find the length of AE.

(ii) Calculate the perimeter of the trapezium $BCDE$.

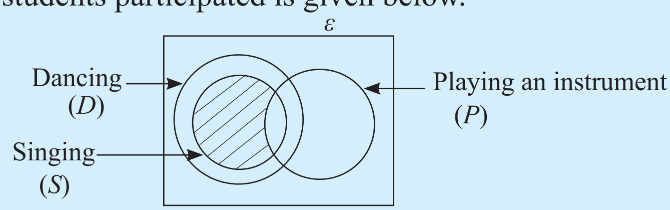
(b) The three cities A, B and C are located such that city B is 50 km from city A, on a bearing of 040° from city A, and city C is directly to the North of city A and on a bearing of 270° from city B.

(i) Draw a suitable sketch and mark the above information in it.

(ii) Find the distance from city A to city C.

(iii) It is necessary to construct a large water tank on top of a concrete pillar to provide water to these three cities. In the above sketch, mark a suitable place where the tank can be built, so that the lengths of the water pipes that carry water from the tank to each of the three cities is the same, and mark this location as T in the sketch.

3. Information on some of the students who were involved in a pageant in which 160 students participated is given below.



$\frac{1}{4}$ of all those who participated in the pageant were involved in at least one of the three activities, dancing, singing and playing an instrument. From the 16 students who were involved in playing an instrument and dancing, 6 students also sang. Twice the number of those who were involved in playing an instrument only was involved in singing and dancing only, and five times the number of those who were involved in playing an instrument only, danced.

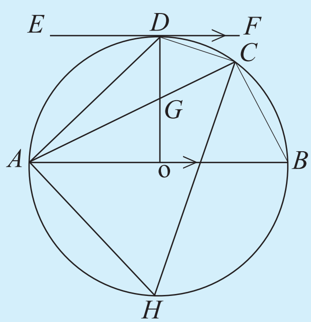
Copy the given Venn diagram in your exercise book and answer the following questions.

(i) Mark the given information accurately in the Venn diagram. How many students were involved in all three activities, singing, dancing and playing an instrument?
(ii) How many were involved in only playing an instrument?
(iii) Express the number that was involved in only one of these three activities as a fraction of the total number of students that participated.
(iv) Describe the activities in which the students in the set represented by $(S' \cap D) \cap P$ (v) were involved. How many students belong to this set?

4. Identical balls of different colours have been placed in two vessels A and B. Vessel A has 3 black balls and 2 white balls. Vessel B has 2 black balls and 3 white balls. A person randomly draws out a ball from vessel A and places it in vessel B. He then randomly draws out a ball from vessel B.

(i) Draw a tree diagram with the probabilities relevant to the above events marked on the branches.
(ii) Using the tree diagram, find the probability of drawing out balls of the same colour on both occasions.

5.



As indicated in the figure, AB is a diameter of the circle with centre O. The tangent EF drawn to the circle at the point D is parallel to AB.

(i) Write down two angles which are equal to $A\hat{B}D$.
(ii) Find the magnitude of $E\hat{D}O$.
(iii) Show that OBCG is a cyclic quadrilateral.

6. Do the following constructions using a pair of compasses and a straight edge with a mm/cm scale and showing the construction lines clearly.

(i) Construct the triangle ABC such that $AB = 8$ cm, $A\hat{B}C = 90^\circ$ and $BC = 4$ cm.

(ii) Construct the trapezium $ABCD$ such that $DC = 2$ cm and DC is parallel to AB.

(iii) Construct the circle that externally touches CB produced at G, CA produced at E and AB produced at F.

ලඝුගණක
மடக்கைகள்
LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	2	4	6	7	9	11	13	15	17
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	2	4	5	7	9	11	12	14	16
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	2	3	5	7	9	10	12	14	15
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	2	3	5	7	8	10	11	13	15
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	2	3	5	6	8	9	11	13	14
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	2	3	5	6	8	9	11	12	14
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	1	3	4	6	7	9	10	12	13
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	1	3	4	6	7	9	10	11	13
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	1	3	4	6	7	8	10	11	12
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	1	3	4	5	7	8	9	11	12
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	1	3	4	5	6	8	9	10	12
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	1	3	4	5	6	8	9	10	11
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	1	2	4	5	6	7	9	10	11
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	1	2	4	5	6	7	8	10	11
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	1	2	3	5	6	7	8	9	10
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	1	2	3	5	6	7	8	9	10
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	1	2	3	4	5	7	8	9	10
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	1	2	3	4	5	6	8	9	10
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	1	2	3	4	5	6	7	8	9
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	1	2	3	4	5	6	7	8	9
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	1	2	3	4	5	6	7	8	9
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	1	2	3	4	5	6	7	8	9
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	1	2	3	4	5	6	7	8	9
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	1	2	3	4	5	6	7	7	8
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	1	2	3	4	5	5	6	7	8
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	1	2	3	4	4	5	6	7	8
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	1	2	3	4	4	5	6	7	8
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Mean Differences heading also printed as: මධ්‍යන්‍ය අන්තරය / இடை வித்தியாசங்கள்

ලඝුගණක
மடக்கைகள்
LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

ප්‍රකෘති සයින
இயற்கைச் சைன்கள்
NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
0	0.0000	0.0029	0.0058	0.0087	0.0116	0.0145	0.0175	89	3	6	9	12	15	17	20	23	26
1	.0175	.0204	.0233	.0262	.0291	.0320	.0349	88	3	6	9	12	15	17	20	23	26
2	.0349	.0378	.0407	.0436	.0465	.0494	.0523	87	3	6	9	12	15	17	20	23	26
3	.0523	.0552	.0581	.0610	.0640	.0669	.0698	86	3	6	9	12	15	17	20	23	26
4	.0698	.0727	.0756	.0785	.0814	.0843	.0872	85	3	6	9	12	15	17	20	23	26
5	0.0872	0.0901	0.0929	0.0958	0.0987	0.1016	0.1045	84	3	6	9	12	14	17	20	23	26
6	.1045	.1074	.1103	.1132	.1161	.1190	.1219	83	3	6	9	12	14	17	20	23	26
7	.1219	.1248	.1276	.1305	.334	.1363	.1392	82	3	6	9	12	14	17	20	23	26
8	.1392	.1421	.1449	.1478	.1507	.1536	.1564	81	3	6	9	11	14	17	20	23	26
9	.1564	.1593	.1622	.1650	.1679	.1708	.1736	80°	3	6	9	11	14	17	20	23	26
10°	0.1736	0.1765	0.1794	0.1822	0.1851	0.1880	0.1908	79	3	6	9	11	14	17	20	23	26
11	.1908	.1937	.1965	.1994	.2022	.2051	.2079	78	3	6	9	11	14	17	20	23	26
12	.2079	.2108	.2136	.2164	.2193	.2221	.2250	77	3	6	9	11	14	17	20	23	26
13	.2250	.2278	.2306	.2334	.2363	.2391	.2419	76	3	6	8	11	14	17	20	23	25
14	.2419	.2447	.2476	.2504	.2532	.2560	.2588	75	3	6	8	11	14	17	20	23	25
15	0.2588	0.2616	0.2644	0.2672	0.2700	0.2728	0.2756	74	3	6	8	11	14	17	20	22	25
16	.2756	.2784	.2812	.2840	.2868	.2896	.2924	73	3	6	8	11	14	17	20	22	25
17	.2924	.2952	.2979	.3007	.3035	.3062	.3090	72	3	6	8	11	14	17	19	22	25
18	.3090	.3118	.3145	.3173	.3201	.3228	.3256	71	3	6	8	11	14	17	19	22	25
19	.3256	.3283	.3311	.3338	.3365	.3393	.3420	70	3	5	8	11	14	16	19	22	25
20°	0.3420	0.3448	0.3475	0.3502	0.3529	0.3557	0.3584	69	3	5	8	11	14	16	19	22	25
21	.3584	.3611	.3638	.3665	.3692	.3719	.3746	68	3	5	8	11	14	16	19	22	24
22	.3746	.3773	.3800	.3827	.3854	.3881	.3907	67	3	5	8	11	13	16	19	21	24
23	.3907	.3934	.3961	.3987	.4014	.4041	.4067	66	3	5	8	11	13	16	19	21	24
24	.4067	.4094	.4120	.4147	.4173	.4200	.4226	65	3	5	8	11	13	16	19	21	24
25	0.4226	0.4253	0.4279	0.4305	0.4331	0.4358	0.4384	64	3	5	8	10	13	16	18	21	24
26	.4348	.4410	.4436	.4462	.4488	.4514	.4540	63	3	5	8	10	13	16	18	21	23
27	.4540	.4566	.4592	.4617	.4643	.4669	.4695	62	3	5	8	10	13	15	18	21	23
28	.4695	.4720	.4746	.4772	.4797	.4823	.4848	61	3	5	8	10	13	15	18	20	23
29	.4848	.4874	.4899	.4924	.4950	.4975	.5000	60°	3	5	8	10	13	15	18	20	23
30°	0.5000	0.5025	0.5050	0.5075	0.5100	0.5125	0.5150	59	3	5	8	10	13	15	18	20	23
31	.5150	.5175	.5200	.5225	.5250	.5275	.5299	58	2	5	7	10	12	15	17	20	22
32	.5299	.5324	.5348	.5373	.5398	.5422	.5446	57	2	5	7	10	12	15	17	20	22
33	.5446	.5471	.5495	.5519	.5544	.5568	.5592	56	2	5	7	10	12	15	17	19	22
34	.5592	.5616	.5640	.5664	.5688	.5712	.5736	55	2	5	7	10	12	14	17	19	22
35	0.5736	0.5760	0.5783	0.5807	0.5831	0.5854	0.5878	54	2	5	7	9	12	14	17	19	21
36	.5878	.5901	.5925	.5948	.5972	.5995	.6018	53	2	5	7	9	12	14	16	19	21
37	.6018	.6041	.6065	.6088	.6111	.6134	.6157	52	2	5	7	9	12	14	16	18	21
38	.6157	.6180	.6202	.6225	.6248	.6271	.6293	51	2	5	7	9	11	14	16	18	20
39	.6293	.6316	.6338	.6361	.6383	.6406	.6428	50°	2	4	7	9	11	13	16	18	20
40°	0.6428	0.6450	0.6472	0.6494	0.6517	0.6539	0.6561	49	2	4	7	9	11	13	15	18	20
41	.6561	.6583	.6604	.6626	.6648	.6670	.6691	48	2	4	7	9	11	13	15	17	20
42	.6691	.6713	.6734	.6756	.6777	.6799	.6820	47	2	4	6	9	11	13	15	17	19
43	.6820	.6841	.6862	.6848	.6905	.6926	.6947	46	2	4	6	8	11	13	15	17	19
44	.6947	.6967	.6988	.7009	.7030	.7050	.7071	45	2	4	6	8	10	12	15	17	19
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

ප්‍රකෘති කෝසයින
இயற்கைக் கோசைன்கள்
NATURAL COSINES

ප්‍රකෘති සයින

இயற்கைச் சைன்கள்

NATURAL SINES

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
45°	0.7071	0.7092	0.7112	0.7133	0.7153	0.7173	0.7193	44°	2	4	6	8	10	12	14	16	18
46	.7193	.7214	.7234	.7254	.7274	.7294	.7314	43	2	4	6	8	10	12	14	16	18
47	.7314	.7333	.7353	.7373	.7392	.7412	.7431	42	2	4	6	8	10	12	14	16	18
48	.7431	.7451	.7470	.7490	.7509	.7528	.7547	41	2	4	6	8	10	12	13	15	17
49	.7547	.7566	.7585	.7604	.7623	.7642	.7660	40°	2	4	6	8	9	11	13	15	17
50°	0.7660	0.7679	0.7698	0.7716	0.7735	0.7753	0.7771	39	2	4	6	7	9	11	13	15	17
51	.7771	.7790	.7808	.7826	.7844	.7862	.7880	38	2	4	5	7	9	11	13	14	16
52	.7880	.7898	.7916	.7934	.7951	.7969	.7986	37	2	4	5	7	9	11	12	14	16
53	.7986	.8004	.8021	.8039	.8056	.8073	.8090	36	2	3	5	7	9	10	12	14	16
54	.8090	.8107	.8124	.8141	.8158	.8175	.8192	35	2	3	5	7	8	10	12	14	15
55	0.8192	0.8208	0.8225	0.8241	0.8258	0.8274	0.8290	34	2	3	5	7	8	10	12	13	15
56	.8290	.8307	.8323	.8339	.8355	.8371	.8387	33	2	3	5	6	8	10	11	13	14
57	.8387	.8403	.8418	.8434	.8450	.8465	.8480	32	2	3	5	6	8	9	11	13	14
58	.8480	.8496	.8511	.8526	.8542	.8557	.8572	31	2	3	5	6	8	9	11	12	14
59	.8572	.8587	.8601	.8616	.8631	.8646	.8660	30°	1	3	4	6	7	9	10	12	13
60°	0.8660	0.8675	0.8689	0.8704	0.8718	0.8732	0.8746	29	1	3	4	6	7	9	10	11	13
61	.8746	.8760	.8774	.8788	.8802	.8816	.8829	28	1	3	4	6	7	8	10	11	12
62	.8829	.8843	.8857	.8870	.8884	.8897	.8910	27	1	3	4	5	7	8	9	11	12
63	.8910	.8923	.8936	.8949	.8962	.8975	.8988	26	1	3	4	5	6	8	9	10	12
64	.8988	.9001	.9013	.9026	.9038	.9051	.9063	25	1	3	4	5	6	8	9	10	11
65	0.9063	0.9075	0.9088	0.9100	0.9112	0.9124	0.9135	24	1	2	4	5	6	7	8	10	11
66	.9135	.9147	.9159	.9171	.9182	.9194	.9205	23	1	2	3	5	6	7	8	9	10
67	.9205	.9216	.9228	.9239	.9250	.9261	.9272	22	1	2	3	4	6	7	8	9	10
68	.9272	.9283	.9293	.9304	.9315	.9325	.9336	21	1	2	3	4	5	6	7	9	10
69	.9336	.9346	.9356	.9367	.9377	.9387	.9397	20	1	2	3	4	5	6	7	8	9
70°	0.9397	0.9407	0.9417	0.9426	0.9436	0.9446	0.9455	19	1	2	3	4	5	6	7	8	9
71	.9455	.9465	.9474	.9483	.9492	.9502	.9511	18	1	2	3	4	5	6	6	7	8
72	.9511	.9520	.9528	.9537	.9546	.9555	.9563	17	1	2	3	4	4	5	6	7	8
73	.9563	.9572	.9580	.9588	.9596	.9605	.9613	16	1	2	2	3	4	5	6	7	7
74	.9613	.9621	.9628	.9636	.9644	.9652	.9659	15	1	2	2	3	4	5	5	6	7
75	0.9659	0.9667	0.9674	0.9681	0.9689	0.9696	0.9703	14	1	1	2	3	4	4	5	6	7
76	.9703	.9710	.9717	.9724	.9730	.9737	.9744	13	1	1	2	3	3	4	5	5	6
77	.9744	.9750	.9757	.9763	.9769	.9775	.9781	12	1	1	2	3	3	4	4	5	6
78	.9781	.9787	.9793	.9799	.9805	.9811	.9816	11	1	1	2	2	3	3	4	5	5
79	.9816	.9822	.9827	.9833	.9838	.9843	.9848	10°	1	1	2	2	3	3	4	4	5
80°	0.9848	0.9853	0.9858	0.9863	0.9868	0.9872	0.9877	9	0	1	1	2	2	3	3	4	4
81	.9877	.9881	.9886	.9890	.9894	.9899	.9903	8	0	1	1	2	2	3	3	3	4
82	.9903	.9907	.9911	.9914	.9918	.9922	.9925	7	0	1	1	2	2	2	3	3	3
83	.9925	.9929	.9932	.9936	.9939	.9942	.9945	6	0	1	1	1	2	2	2	3	3
84	.9945	.9948	.9951	.9954	.9957	.9959	.9962	5	0	1	1	1	1	2	2	2	3
85	0.9962	0.9964	0.9967	0.9969	0.9971	0.9974	0.9976	4									
86	.9976	.9978	.9980	.9981	.9983	.9985	.9986	3									
87	.9986	.9988	.9989	.9990	.9992	.9993	.9994	2	(අන්තරය ඉතා කුඩා බැවින් වගු ගත කිරීම අනවශ්‍යය.)								
88	.9994	.9995	.9996	.9997	.9997	.9998	.9998	1									
89	0.9998	0.9999	0.9999	1.0000	1.0000	1.0000	1.0000	0°									
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

ප්‍රකෘති කෝසයින

இயற்கைக் கோசைன்கள்

NATURAL COSINES

ප්‍රකෘති ටැංජන

இயற்கைத் தான்சன்கள்

NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									මධ්‍යන්‍ය අන්තරය இடை வித்தியாசங்கள் Mean Differences								
0	0.0000	0.0029	0.0058	0.0087	0.0116	0.0145	0.0175	89°	3	6	9	12	15	17	20	23	26
1	.0175	.0204	.0233	.0262	.0291	.0320	.0349	88	3	6	9	12	15	17	20	23	26
2	.0349	.0378	.0407	.0437	.0466	.0495	.0524	87	3	6	9	12	15	18	20	23	26
3	.0524	.0553	.0582	.0612	.0641	.0670	.0699	86	3	6	9	12	15	18	20	23	26
4	.0699	.0729	.0758	.0787	.0816	.0846	.0875	85	3	6	9	12	15	18	21	23	26
5	0.0875	0.0904	0.0934	0.0963	0.0992	0.1022	0.1051	84	3	6	9	12	15	18	21	24	26
6	.1051	.1080	.1110	.1139	.1169	.1198	.1228	83	3	6	9	12	15	18	21	24	27
7	.1228	.1257	.1287	.1317	.1346	.1376	.1405	82	3	6	9	12	15	18	21	24	27
8	.1405	.1435	.1465	.1495	.1524	.1554	.1584	81	3	6	9	12	15	18	21	24	27
9	.1584	.1614	.1644	.1673	.1703	.1733	.1763	80°	3	6	9	12	15	18	21	24	27
10°	0.1763	0.1793	0.1823	0.1853	0.1883	0.1914	0.1944	79	3	6	9	12	15	18	21	24	27
11	.1944	.1974	.2004	.2035	.2065	0.2095	0.2126	78	3	6	9	12	15	18	21	24	27
12	.2126	.2156	.2186	.2217	.2247	.2278	.2309	77	3	6	9	12	15	18	21	24	27
13	.2309	.2339	.2370	.2401	.2432	.2462	.2493	76	3	6	9	12	15	18	22	25	28
14	.2493	.2524	.2555	.2586	.2617	.2648	.2679	75	3	6	9	12	16	19	22	25	28
15	0.2679	0.2711	0.2742	0.2773	0.2805	0.2836	0.2867	74	3	6	9	13	16	19	22	25	28
16	.2867	.2899	.2931	.2962	.2994	.3026	.3057	73	3	6	9	13	16	19	22	25	28
17	.3057	.3089	.3121	.3153	.3185	.3217	.3249	72	3	6	10	13	16	19	22	26	29
18	.3249	.3281	.3314	.3346	.3378	.3411	.3443	71	3	6	10	13	16	19	23	26	29
19	.3443	.3476	.3508	.3541	.3574	.3607	.3640	70°	3	7	10	13	16	20	23	26	29
20°	0.3640	0.3673	0.3706	0.3739	0.3772	0.3805	0.3839	69	3	7	10	13	17	20	23	27	30
21	.3839	.3872	.3906	.3939	.3973	.4006	.4040	68	3	7	10	13	17	20	24	27	30
22	.4040	.4074	.4108	.4142	.4176	.4210	.4245	67	3	7	10	14	17	20	24	27	31
23	.4245	.4279	.4314	.4348	.4383	.4417	.4452	66	3	7	10	14	17	21	24	28	31
24	.4452	.4487	.4522	.4557	.4592	.4628	.4663	65	4	7	11	14	18	21	25	28	32
25	0.4663	0.4699	0.4734	0.4770	0.4806	0.4841	0.4877	64	4	7	11	14	18	21	25	29	32
26	.4877	.4913	.4950	.4986	.5022	.5059	.5095	63	4	7	11	15	18	22	25	29	33
27	.5095	.5132	.5169	.5206	.5243	.5280	.5317	62	4	7	11	15	18	22	26	30	33
28	.5317	.5354	.5392	.5430	.5467	.5505	.5543	61	4	8	11	15	19	23	26	30	34
29	.5543	.5581	.5619	.5658	.5696	.5735	.5774	60°	4	8	12	15	19	23	27	31	35
30°	0.5774	0.5812	0.5851	0.5890	0.5930	0.5969	0.6009	59	4	8	12	16	20	24	27	31	35
31	.6009	.6048	.6088	.6128	.6168	.6208	.6249	58	4	8	12	16	20	24	28	32	36
32	.6249	.6289	.6330	.6371	.6412	.6453	.6494	57	4	8	12	16	20	25	29	33	37
33	.6494	.6536	.6577	.6619	.6661	.6703	.6745	56	4	8	13	17	21	25	29	33	38
34	.6745	.6787	.6830	.6873	.6916	.6959	.7002	55	4	9	13	17	21	26	30	34	39
35	0.7002	0.7046	0.7089	0.7133	0.7177	0.7221	0.7265	54	4	9	13	18	22	26	31	35	40
36	.7265	.7310	.7355	.7400	.7445	.7490	.7536	53	5	9	14	18	23	27	32	36	41
37	.7536	.7581	.7627	.7673	.7720	.7766	.7813	52	5	9	14	19	23	28	32	37	42
38	.7813	.7860	.7907	.7954	.8002	.8050	.8098	51	5	10	14	19	24	29	33	38	43
39	.8098	.8146	.8195	.8243	.8292	.8342	.8391	50°	5	10	15	20	24	29	34	39	44
40°	0.8391	0.8441	0.8491	0.8541	0.8591	0.8642	0.8693	49	5	10	15	20	25	30	35	40	45
41	.8693	.8744	.8796	.8847	.8899	.8952	.9004	48	5	10	16	21	26	31	36	41	47
42	.9004	.9057	.9110	.9163	.9217	.9271	.9325	47	5	11	16	21	27	32	37	43	48
43	.9325	.9380	.9435	.9490	.9545	.9601	.9657	46	6	11	17	22	28	33	39	44	50
44	.9657	.9731	.9770	.9827	.9884	.9942	1.0000	45	6	11	17	23	29	34	40	46	51
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

ප්‍රකෘති කෝටැංජන

இயற்கைக் கோதான்சன்கள்

NATURAL COTANGENTS

ප්‍රකෘති ටැංජන
இயற்கைத் தான்சன்கள்
NATURAL TANGENTS

	0′	10′	20′	30′	40′	50′	60′		1′	2′	3′	4′	5′	6′	7′	8′	9′
									මධ්‍යන්‍ය අන්තරය / இடை வித்தியாசங்கள் / Mean Differences								
45°	1.0000	1.0058	1.0117	1.0176	1.0235	1.0295	1.0355	44°	6	12	18	24	30	36	41	47	53
46	.0355	.0416	.0477	.0538	.0599	.0661	.0724	43	6	12	18	25	31	37	43	49	55
47	.0724	.0786	.0850	.0913	.0977	.1041	.1106	42	6	13	19	26	32	38	45	51	57
48	.1106	.1171	.1237	.1303	.1369	.1436	.1504	41	7	13	20	27	33	40	46	53	60
49	.1504	.1571	.1640	.1708	.1778	.1847	.1918	40°	7	14	21	28	34	41	48	55	62
50°	1.1918	1.1988	1.2059	1.2131	1.2203	1.2276	1.2349	39	7	14	22	29	36	43	50	58	65
51	.2349	.2423	.2497	.2572	.2647	.2723	.2799	38	8	15	23	30	38	45	53	60	68
52	.2799	.2876	.2954	.3032	.3111	.3190	.3270	37	8	16	24	31	39	47	55	63	71
53	.3270	.3351	.3432	.3514	.3597	.3680	.3764	36	8	16	25	33	41	49	58	66	74
54	.3764	.3848	.3934	.4019	.4106	.4193	.4281	35	9	17	26	35	43	52	60	69	78
55	1.4281	1.4370	1.4460	1.4550	1.4641	1.4733	1.4826	34	9	18	27	36	45	54	63	73	82
56	.4826	.4919	.5013	.5108	.5204	.5301	.5399	33	10	19	29	38	48	57	67	76	86
57	.5399	.5497	.5597	.5697	.5798	.5900	.6003	32	10	20	30	40	50	60	71	81	91
58	.6003	.6107	.6212	.6319	.6426	.6534	.6643	31	11	21	32	43	53	64	75	85	96
59	.6643	.6753	.6864	.6977	.7090	.7205	.7321	30°	11	23	34	45	56	68	79	90	102
60°	1.732	1.744	1.756	1.767	1.780	1.792	1.804	29	1	2	4	5	6	7	8	10	11
61	1.804	1.816	1.829	1.842	1.855	1.868	1.881	28	1	3	4	5	6	8	9	10	12
62	1.881	1.894	1.907	1.921	1.935	1.949	1.963	27	1	3	4	5	7	8	10	11	12
63	1.963	1.977	1.991	2.006	2.020	2.035	2.050	26	1	3	4	6	7	9	10	12	13
64	2.050	2.066	2.081	2.097	2.112	2.128	2.145	25	2	3	5	6	8	9	11	13	14
65	2.145	2.161	2.177	2.194	2.211	2.229	2.246	24	2	3	5	7	8	10	12	14	15
66	2.246	2.264	2.282	2.300	2.318	2.337	2.356	23	2	4	5	7	9	11	13	15	16
67	2.356	2.375	2.394	2.414	2.434	2.455	2.475	22	2	4	6	8	10	12	14	16	18
68	2.475	2.496	2.517	2.539	2.560	2.583	2.605	21	2	4	6	9	11	13	15	17	20
69	2.605	2.628	2.651	2.675	2.699	2.723	2.747	20°	2	5	7	9	12	14	17	19	21
70°	2.747	2.773	2.798	2.824	2.850	2.877	2.904	19	3	5	8	10	13	16	18	21	23
71	2.904	2.932	2.960	2.989	3.018	3.047	3.078	18	3	6	9	12	14	17	20	23	26
72	3.078	3.108	3.140	3.172	3.204	3.237	3.271	17	3	6	10	13	16	19	23	26	29
73	3.271	3.305	3.340	3.376	3.412	3.450	3.487	16	4	7	11	14	18	22	25	29	32
74	3.487	3.526	3.566	3.606	3.647	3.689	3.732	15	4	8	12	16	20	24	29	33	37
75	3.732	3.776	3.821	3.867	3.914	3.962	4.011	14	5	9	14	19	23	28	33	37	42
76	4.011	4.061	4.113	4.165	4.219	4.275	4.331	13	5	11	16	21	27	32	37	43	48
77	4.331	4.390	4.449	4.511	4.574	4.638	4.705	12	6	12	19	25	31	37	44	50	56
78	4.705	4.773	4.843	4.915	4.989	5.066	5.145	11	7	15	22	29	37	44	51	59	66
79	5.145	5.226	5.309	5.396	5.485	5.576	5.671	10°	9	18	26	35	44	53	61	70	79
80°	5.671	5.769	5.871	5.976	6.084	6.197	6.314	9									
81	6.314	6.435	6.561	6.691	6.827	6.968	7.115	8									
82	7.115	7.269	7.429	7.596	7.770	7.953	8.144	7	අන්තර ශීඝ්‍රයෙන් වෙනස් වෙයි								
83	8.144	8.345	8.556	8.777	9.010	9.255	9.514	6									
84	9.514	9.788	10.078	10.385	10.712	11.059	11.430	5	வித்தியாசங்கள் விரைந்து மாறுவன								
85	11.43	11.83	12.25	12.71	13.20	13.73	14.30	4									
86	14.30	14.92	15.60	16.35	17.17	18.07	19.08	3	Differences change rapidly								
87	19.08	20.21	21.47	22.90	24.54	26.43	28.64	2									
88	28.64	31.24	34.37	38.19	42.96	49.10	57.29	1									
89	57.29	68.75	85.94	114.59	171.89	343.77	∞	0°									
	60′	50′	40′	30′	20′	10′	0′		1′	2′	3′	4′	5′	6′	7′	8′	9′

ප්‍රකෘති කෝටැංජන
இயற்கைக் கோதான்சன்கள்
NATURAL COTANGENTS

Glossary

A

Adjacent side	බද්ධ පාදය	அயற் பக்கம்
Angles in the same segment	ඒකාන්තර වෘත්ත ඛණ්ඩයේ කෝණ	ஒரேதுண்டக் கோணங்கள்

C

Centre	කේන්ද්‍රය	மையம்
Chord	ජ්‍යාය	நாண்
Circle	වෘත්තය	வட்டம்
Circumcircle	පරිවෘත්තය	சுற்று வட்டம்
Column matrix	තීර න්‍යාසය	நிரற் தாயம்
Cosine	කෝසයිනය	கோசைன்
Cyclic quadrilateral	වෘත්ත චතුරස්‍ර	வட்ட நாற்பக்கல்

D

Dependent Events	පරායත්ත සිද්ධි	சார் நிகழ்ச்சி

E

Element	අවයව	மூலகம்
Elements of a matrix	න්‍යාසයක අවයව	தாயமொன்றின் மூலகங்கள்
Exterior angle	බාහිර කෝණය	புறக்கோணம்
Exterior point	බාහිර ලක්ෂ්‍යය	புறப்புள்ளி
Excircle/Escribed circle	බාහිර් වෘත්තය	வெளி வட்டம்

G

Grid	කොටු දැල	நெய்யரி

H

Hypotenuse	කර්ණය	செம்பக்கம்

I

Independent events	ස්වායත්ත සිද්ධි	சாரா நிகழ்ச்சிகள்
Inequalities	අසමානතා	சமனிலிகள்
Inscribed circle/ Incircle	අන්තර්වෘත්තය	உள்வட்டம்
Interior opposite angle	අභ්‍යන්තර සම්මුඛ කෝණය	அகத்தெதிர் கோணம்
Intersection of sets	කුලක ඡේදනය	தொடைகளின் இடைவெட்டு

L		
Locus	පථය	ஒழுக்கு
M		
Matrices	න්‍යාස	தாயங்கள்
O		
Opposite angles	සම්මුඛ කෝණ	எதிர்க் கோணங்கள்
Opposite side	සම්මුඛ පාදය	எதிர்ப் பக்கங்கள்
Order of a matrix	න්‍යාසයේ ගණය	தாயத்தின் வரிசை
P		
Perpendicular	ලම්බකය	செங்குத்து
Point	ලක්ෂ්‍යය	புள்ளி
Pythagoras' theorem	පයිතගරස් ප්‍රමේයය	பைதகரசின் தேற்றம்
Pythagoras' triple	පයිතගරස් ත්‍රික	பைதகரசின் மும்மை
R		
Radius	අරය	ஆரை
Random Experiments	සසම්භාවී පරීක්ෂණ	எழுமாற்றுப் பரிசோதனை
Riders	අනුමේයයන්	ஏறிகள்
Right angled triangles	සෘජුකෝණික ත්‍රිකෝණ	செங்கோண முக்கோணம்
Row matrix	පේළි න්‍යාසය	நிரைத் தாயம்
S		
Sample space	නියැදි අවකාශය	மாதிரிவெளி
Segment of a circle	වෘත්ත ඛණ්ඩය	வட்டத்துத்துண்டம்
Set	කුලකය	தொடை
Sine	සයිනය	சைன்
Solution set	විසඳුම් කුලකය	தீர்வுத் தொடை
Square matrix	සමචතුරස්‍ර න්‍යාසය	சதுரத் தாயம்
Subtended	ආපාතිත	எதிரமை
Supplementary	පරිපූරක	மிகை நிரப்புகின்ற
Symmetric matrix	සමමිති න්‍යාසය	சமச்சீர்த் தாயம்

T

Tangent	ස්පර්ශකය	தொடலி
Tree diagram	රුක් සටහන	மரவரிப்படம்
Trigonometric ratios	ත්‍රිකෝණමිතික අනුපාත	திரிகோண கணித விகிதங்கள்
Trignometry	ත්‍රිකෝණමිතිය	திரிகோண கணிதம்

U

Union of sets	කුලක මේලය	தொடைகளின் ஒன்றிப்பு
Unit matrix	ඒකක න්‍යාසය	அலகுத் தாயம்

V

Venn diagram	වෙන් රූපය	வென் வரிப்படம்

Sequence of the Lessons

Chapter of Textbook	No.of Periods
1 Term	
1. Real Numbers	10
2. Indices and Logarithms I	08
3. Indices and Logarithms II	06
4. Surface Area of Solids	05
5. Volume of the Solids	05
6. Binomial Expressions	04
7. Algebraic Fractions	04
8. Areas of Plane Figures between Parallel Lines	12
2 Term	
09. Percentages	06
10. Share Market	05
11. Mid Point Theorem	05
12. Graphs	12
13. Formulae	10
14. Equiangular Triangles	12
15. Data representaion and Interpretation	12
16. Geometric Progressions	06
3 Term	
17. Pythagoras's Theorem	04
18. Trignometry	12
19. Matrices	08
20. Inequalities	06
21. Cyclic Quadrilaterals	10
22. Tangent	10
23. Constructions	05
24. Sets	06
25. Probability	07