අධ්‍යයන පොදු සහතික පත්‍ර (උසස් පෙළ) විභාගය, 2021(2022)
கல்விப் பொதுத் தராதரப் பத்திர (உயர் தர)ப் பரீட்சை, 2021(2022)
General Certificate of Education (Adv. Level) Examination, 2021(2022)

සංයුක්ත ගණිතය I
இணைந்த கணிதம் I
Combined Mathematics I

10 E I

Part B

※ Answer **five** questions only.

11. (*a*) Let $k>1$. Show that the equation $x^2-2(k+1)x+(k-3)^2=0$ has real distinct roots. Let α and β be these roots. Write down $\alpha+\beta$ and $\alpha\beta$ in terms of k, and find the values of k such that both α and β are positive.
Now, let $1<k<3$. Find the quadratic equation whose roots are $\dfrac{1}{\sqrt{\alpha}}$ and $\dfrac{1}{\sqrt{\beta}}$, in terms of k.

(*b*) Let $f(x)=2x^3+ax^2+bx+1$ and $g(x)=x^3+cx^2+ax+1$, where $a, b, c\in\mathbb{R}$. It is given that the remainder when $f(x)$ is divided by $(x-1)$ is 5, and that the remainder when $g(x)$ is divided by x^2+x-2 is $x+1$. Find the values of a, b and c.
Also, with these values for a, b and c, show that $f(x)-2g(x)\leq\dfrac{13}{12}$ for all $x\in\mathbb{R}$.

12. (*a*) It is required to form a 4-digit number consisting of 4 digits taken from the 10 digits given below:

1, 1, 1, 2, 2, 3, 3, 4, 5, 5

Find the number of different such 4-digit numbers that can be formed

(i) if all 4 digits chosen are different,

(ii) if any 4 digits can be chosen.

(*b*) Let $U_r=\dfrac{-16r^3+12r^2+40r+9}{5(2r+1)^2(2r-1)^2}$ for $r\in\mathbb{Z}^+$.

Determine the values of the real constants A and B such that $U_r=\dfrac{A(r-1)}{(2r+1)^2}-\dfrac{(r-B)}{(2r-1)^2}$ for $r\in\mathbb{Z}^+$.

Hence, find $f(r)$ such that $\dfrac{1}{5^{r-1}}U_r=f(r)-f(r-1)$ for $r\in\mathbb{Z}^+$, and

show that $\displaystyle\sum_{r=1}^{n}\frac{1}{5^{r-1}}U_r=1+\frac{n-1}{5^n(2n+1)^2}$ for $n\in\mathbb{Z}^+$.

Deduce that the infinite series $\displaystyle\sum_{r=1}^{\infty}\frac{1}{5^{r-1}}U_r$ is convergent and find its sum.

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13. (*a*) Let $\mathbf{A} = \begin{pmatrix} a & 0 & 3 \\ 0 & a & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} a & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$, where $a \in \mathbb{R}$.

Also, let $\mathbf{C} = \mathbf{AB}^{\mathrm{T}}$. Find $\mathbf{C}$ in terms of a, and show that $\mathbf{C}^{-1}$ exists for all $a \neq 0$.

Write down $\mathbf{C}^{-1}$ in terms of a, when it exists.

Show that if $\mathbf{C}^{-1}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 9 \\ -11 \end{pmatrix}$, then $a = 2$.

With this value for a, find the matrix $\mathbf{D}$ such that $\mathbf{DC} - \mathbf{C}^{\mathrm{T}}\mathbf{C} = 8\mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

(*b*) Let $z_1 = 1 + \sqrt{3}i$ and $z_2 = 1 + i$. Express $\frac{z_1}{z_2}$ in the form $x + iy$, where $x, y \in \mathbb{R}$.

Also, express each of the complex numbers z_1 and z_2 in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $0 < \theta < \frac{\pi}{2}$, and **hence**, show that $\frac{z_1}{z_2} = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$.

Deduce that $\cos\left(\frac{\pi}{12}\right) = \frac{1+\sqrt{3}}{2\sqrt{2}}$.

(*c*) Let $n \in \mathbb{Z}^+$ and $\theta \neq 2k\pi \pm \frac{\pi}{2}$ for $k \in \mathbb{Z}$.

Using De Moivre's theorem, show that $(1 + i\tan\theta)^n = \sec^n\theta(\cos n\theta + i\sin n\theta)$.

Hence, obtain a similar expression for $(1 - i\tan\theta)^n$, and

show that $(1 + i\tan\theta)^n + (1 - i\tan\theta)^n = 2\sec^n\theta\cos n\theta$.

Deduce that $z = i\tan\left(\frac{\pi}{10}\right)$ is a solution of $(1+z)^{25} + (1-z)^{25} = 0$.

14. (*a*) Let $f(x) = \frac{4x+1}{x(x-2)}$ for $x \neq 0, 2$.

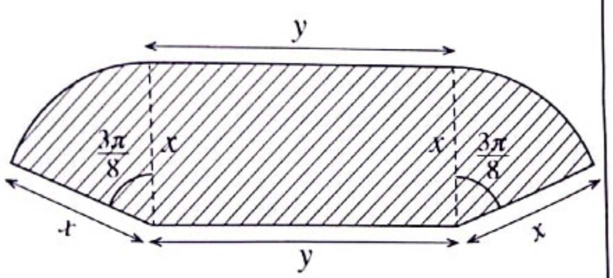
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = -\frac{2(2x-1)(x+1)}{x^2(x-2)^2}$ for $x \neq 0, 2$.

Hence, find the intervals on which $f(x)$ is increasing and the intervals on which $f(x)$ is decreasing.

Sketch the graph of $y = f(x)$ indicating the asymptotes, x-intercept and the turning points.

Using this graph, find all real values of x satisfying the inequality $f(x) + |f(x)| > 0$.

(*b*) The shaded region S of the adjoining figure shows a garden consisting of a rectangle and two sectors of a circle each subtending an angle $\frac{3\pi}{8}$ at the centre. Its dimensions, in metres, are shown in the figure. The area of S is given to be $36\ \mathrm{m}^2$. Show that the perimeter p m of S is given by $p = 2x + \frac{72}{x}$ for $x > 0$ and that p is minimum when $x = 6$.



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15. (*a*) Find the values of the constants A, B and C such that

$$x^4 + 3x^3 + 4x^2 + 3x + 1 = A(x^2 + 1)^2 + Bx(x^2 + 1) + Cx^2 \text{ for all } x \in \mathbb{R}.$$

Hence, write down $\dfrac{x^4+3x^3+4x^2+3x+1}{x(x^2+1)^2}$ in partial fractions and

find $\displaystyle\int \frac{x^4+3x^3+4x^2+3x+1}{x(x^2+1)^2}\,dx$.

(*b*) Let $I = \displaystyle\int_0^{\frac{1}{4}} \sin^{-1}\left(\sqrt{x}\right)dx$. Show that $I = \dfrac{\pi}{24} - \dfrac{1}{2}\displaystyle\int_0^{\frac{1}{4}} \sqrt{\frac{x}{1-x}}\,dx$ and **hence**, evaluate I.

(*c*) Show that $\dfrac{d}{dx}\left(x\ln(x^2+1) + 2\tan^{-1}x - 2x\right) = \ln(x^2+1)$.

Hence, find $\displaystyle\int \ln(x^2+1)\,dx$ and show that $\displaystyle\int_0^1 \ln(x^2+1)\,dx = \frac{1}{2}(\ln 4 + \pi - 4)$.

Using the result $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, where a is a constant,

find the value of $\displaystyle\int_0^1 \ln\left[(x^2+1)(x^2-2x+2)\right]dx$.

16. Let $P \equiv (x_1, y_1)$ and l be the straight line given by $ax + by + c = 0$. Show that the coordinates of any point on the line through the point P and perpendicular to l are given by $(x_1 + at, y_1 + bt)$, where $t \in \mathbb{R}$.

Deduce that the perpendicular distance from P to l is $\dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$.

Let l be the straight line $x + y - 2 = 0$. Show that the points $A \equiv (0, 6)$ and $B \equiv (3, -3)$ lie on opposite sides of l.

Find the acute angle between l and the line AB.

Find the equations of the circles S_1 and S_2 with centres at A and B, respectively, and touching l.

Let C be the point of intersection of l and the line AB. Find the coordinates of the point C.

Find also the equation of the other common tangent through C to S_1 and S_2.

Show that the equation of the circle that passes through the origin, bisects the circumference of S_1 and orthogonal to S_2 is $3x^2 + 3y^2 - 38x - 22y = 0$.

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17. (*a*) Write down $\cos(A+B)$ and $\cos(A-B)$ in terms of $\cos A$, $\cos B$, $\sin A$ and $\sin B$.

Hence, show that $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

Deduce that $\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$.

Solve the equation $\cos 9x + \cos 7x + \cot x\,(\cos 9x - \cos 7x) = 0$.

(*b*) In the usual notation, state and prove the **Cosine Rule** for a triangle ABC.

Let $x \neq n\pi + \frac{\pi}{2}$ for $n \in \mathbb{Z}$. Show that $\sin 2x = \frac{2\tan x}{1+\tan^2 x}$.

In a triangle ABC, it is given that $AB = 20$ cm, $BC = 10$ cm and $\sin 2B = \frac{24}{25}$.

Show that there are two distinct such triangles and find the length of AC for each.

(*c*) Solve the equation $\sin^{-1}\left[(1+e^{-2x})^{-\frac{1}{2}}\right] + \tan^{-1}(e^{x}) = \tan^{-1}(2)$.

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